A POLYAKOV FORMULA FOR SECTORS

CLARA L. ALDANA AND JULIE ROWLETT

ABSTRACT. We consider finite area convex Euclidean circular sectors. We prove a variational Polyakov formula which shows how the zeta-regularized determinant of the Laplacian varies with respect to the opening angle. Varying the angle corresponds to a conformal deformation in the direction of a conformal factor with a logarithmic singularity at the corner. As an application of the method, we obtain an *analogue* Polyakov formula for a surface with one conical singularity. We compute the zeta-regularized determinant of rectangular domains of fixed area and prove that it is uniquely maximized by the square.

1. Introduction

Polyakov's formula expresses a difference of zeta-regularized determinants of Laplace operators, an anomaly of global quantities, in terms of simple local quantities. The main applications of Polyakov's formula are in differential geometry and mathematical physics. In mathematical physics, this formula arose in the study of the quantum theory of strings [32] and has been used in connection to conformal quantum field theory [6] and Feynmann path integrals [16].

In differential geometry, Polyakov's formula was used in the work of Osgood, Phillips and Sarnak [30] to prove that under certain restrictions on the Riemannian metric, the determinant is maximized at the uniform metric inside a conformal class. Their result holds for smooth closed surfaces and for surfaces with smooth boundary. This result was generalized to surfaces with cusps and funnel ends in [2]. The techniques used in this article are similar to the ones used by the first author in [3] to prove a Polyakov formula for the relative determinant for surfaces with cusps.

We expect that the formula of Polyakov we shall demonstrate here will have applications to differential geometry in the spirit of [30]. Our formula is a step towards answering some of the many open questions for domains with corners such as polygonal domains and surfaces with conical singularities: what are suitable restrictions to have an extremal of the determinant in a conformal class as in [30]? Will it be unique? Does the regular n-gon maximize the determinant on all n-gons of fixed area? What happens to the determinant on a family of n-gons which collapses to a segment?

1.1. **The zeta regularized determinant of the Laplacian.**

Consider a smooth n-dimensional manifold M with Riemannian metric g. We denote by Δ_g the Laplace operator associated to the metric g. We consider the positive Laplacian $\Delta_g \geq 0$. If M is compact and without boundary, or if M has non-empty boundary and suitable

C.L. Aldana was supported by ANR grant ACG: ANR-10-BLAN 0105 and by the Fonds National de la Recherche, Luxembourg 7926179.

boundary conditions are imposed, then the eigenvalues of the Laplace operator form an increasing, discrete subset of $\mathbb{R}^+$,

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots .$$

These eigenvalues tend toward infinity according to Weyl's law [38],

$$\lambda_k^{\frac{n}{2}} \sim \frac{(2\pi)^n k}{\omega_n \mathrm{Vol}(M)}, \quad \text{as } k \to \infty,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Ray and Singer generalized the notion of determinant of matrices to the Laplace-de Rham operator on forms using an associated zeta function [33]. The spectral zeta function associated to the Laplace operator is defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{n}{2}$ by

$$\zeta(s) := \sum_{\lambda_k > 0} \lambda_k^{-s}.$$

By Weyl's law, the zeta function is holomorphic on the half-plane $\{\mathrm{Re}(s) > n/2\}$, and it is well known that the heat equation can be used to prove that the zeta function admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$ [33]. Consequently, the zeta-regularized determinant of the Laplace operator may be defined as

$$\det(\Delta) := e^{-\zeta'(0)}. \tag{1.1}$$

In this way, the determinant of the Laplacian is a number that depends only on the spectrum; it is a spectral invariant. Furthermore, it is also a global invariant, meaning that in general it can not be expressed as an integral over the manifold of local quantities.

1.2. **Polyakov's formula for smooth surfaces.** Let (M, g) be a smooth Riemannian surface. Let $g_t = e^{2\sigma(t)} g$ be a one-parameter family of metrics in the conformal class of g depending smoothly on $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$. Assume that each conformal factor $\sigma(t)$ is a smooth function on M. The Laplacian for the metric g_t relates to the Laplacian of the metric g via

$$\Delta_{g_t} = e^{-2\sigma(t)} \Delta_g.$$

The variation of the Laplacian for the metric g_t with respect to the parameter t is

$$\partial_t \Delta_{g_t}|_{t=0} = -2\sigma'(0) \Delta_{g_0}, \quad g_0 = e^{2\sigma(0)} g. \tag{1.2}$$

In this setting, Polyakov's formula gives the variation of the determinant of the family of conformal Laplacians Δ_{g_t} with respect to the parameter t of the *conformal factor* $\sigma(t)$, [19], [2]

$$\partial_t \log \det(\Delta_{g_t}) = -\frac{1}{24\pi} \int_M \sigma'(t) \ \mathrm{Scal}_t \ dA_{g_t} + \partial_t \log \mathrm{Area}(M, g_t), \tag{1.3}$$

where Scal_t denotes the scalar curvature of the metric g_t. This is the type of formula that we demonstrate here and may refer to it as either the differentiated or variational Polyakov formula or simply Polyakov's formula. The classical form of Polyakov's formula is the "integrated form" which expresses the determinant as an anomaly; for a surface M with smooth boundary it was first proven by Alvarez [4]; see also [30]. There are two main difficulties which distinguish our work from the case of closed surfaces: (1) the presence of a geometric singularity in the domain or surface and (2) the presence of an analytic singularity in the conformal factor.

1.3. **Conical singularities.** Analytically and geometrically, the presence of even the simplest conical singularity, a corner in a Euclidean domain, has a profound impact on the Laplace operator. As in the case of a manifold with boundary, the Laplace operator is not essentially self-adjoint. It has many self adjoint extensions, and the spectrum depends on the choice of self-adjoint extension. Thus, the zeta-regularized determinant of the Laplacian also depends upon this choice [28]. In addition, conical singularities add regularity problems that do not appear when the boundary of the domain or manifold is smooth.

In recent years there has been progress towards understanding the behavior of the determinant of certain self-adjoint extensions of the Laplace operator, most notably the Friedrichs extension, on surfaces with conical singularities. This progress represents different aspects that have been studied by Kokotov [18], Hillairet and Kokotov [17], Loya et al [23], Spreafico [35], and Sher [34]. In particular, the results by Aurell and Salomonson in [5] inspired our present work. Using heuristic arguments they computed a formula for the contribution of the corners to the variation of the determinant on a polygon [5, eqn (51)]. Here we use modern techniques to rigorously prove the differentiated Polyakov formula for an angular sector. Our work is complementary to those mentioned above since the dependence of the determinant of the Friedrichs extension of the Laplacian with respect to changes of the cone angle has not been addressed previously. In addition, our formula can be related to a variational principle.

1.4. **Organization and main results.** In §2, we develop the requisite geometric and analytic tools needed to prove our first main result.

Theorem 1 (Polyakov formula for sectors)**.** *Let $\{S_\gamma\}_{\gamma\in(0,\pi)}$ be a family of finite circular sectors in $\mathbb{R}^2$, where S_γ has opening angle γ and unit radius. Let Δ_γ be the Euclidean Dirichlet Laplacian on S_γ. Then for any $\alpha\in(0,\pi)$*

$$\frac{\partial}{\partial\gamma}\big(-\log(\det(\Delta_\gamma))\big)\Big|_{\gamma=\alpha} = \mathrm{fp}_{t=0}\int_{S_\alpha}\frac{2}{\alpha}(1+\log(r))H_{S_\alpha}(t,r,\phi,r,\phi)r\,dr\,d\phi, \tag{1.4}$$

where $\mathrm{fp}_{t=0}$ denotes the finite part (Hadamard's partie finie) of the integral at $t=0$, and H_{S_α} denotes the heat kernel on the finite angular sector S_α.

If the radial direction is multiplied by a factor of R, which is equivalent to scaling the metrics by R^2, the determinant of the Laplacian transforms as

$$\det(\Delta_\alpha)\mapsto R^{-2\zeta_{\Delta_\alpha}(0)}\det(\Delta_\alpha).$$

Notice that equation (1.4) can also be written as

$$\begin{aligned}\frac{\partial}{\partial\gamma}\big(-\log(\det(\Delta_\gamma))\big)\Big|_{\gamma=\alpha} &= \frac{2}{\alpha}\zeta_{\Delta_\alpha}(0)\\ &\quad+\mathrm{fp}_{t=0}\int_{S_\alpha}\frac{2}{\alpha}\log(r)H_{S_\alpha}(t,r,\phi,r,\phi)r\,dr\,d\phi,\end{aligned} \tag{1.5}$$

where for the finite sector S_α, $\zeta_{\Delta_\alpha}(0)=\frac{\alpha}{12\pi}+\frac{\pi^2-\alpha^2}{24\pi\alpha}+\frac{1}{8}$ cf. [27, eq. (1.4)].

The proof of these results comprises §3. In §4, we outline the proof of the existence and finiteness of the finite parts of the integrals in Theorem 1. This general argument is carried out in detail for the finite sector of opening angle $\pi/2$.

Theorem 2. *Let $S_{\pi/2} \subset \mathbb{R}^2$ be a circular sector of opening angle $\pi/2$ and radius one. Then the contribution of the corner at the origin in equation (1.4) is given by*

$$-\frac{1}{4\pi} - \frac{\gamma_e}{4\pi},$$

where γ_e is the Euler-Mascheroni constant. Moreover,

$$\frac{\partial}{\partial\gamma}\big(- \log(\det(\Delta_\gamma))\big)\Big|_{\gamma=\pi/2} = \frac{1}{6\pi} - \frac{3}{4\pi} - \frac{3\gamma_e}{4\pi}.$$

The proof of Theorem 2 is contained in §4 and illustrates the method we shall use to compute the general case of a sector of opening angle $\alpha \in (0, \pi)$. Since the general case will require several additional lengthy calculations, we shall carry it out in forthcoming work.

The same tools used to prove Theorem 1 allow us to obtain a similar (though weaker) result for a surface with a conical singularity.

Proposition 1 (Polyakov formula for surfaces with one conical singularity)**.** *Let (M, g) be a Riemannian surface with an isolated conical singularity of opening angle $\alpha \in (0, \pi)$. Let $\{h_\gamma = e^{2\sigma(\gamma)} g\}$ denote a one-parameter family of conformal metrics satisfying Definition 2. Let*

$$\delta\sigma_\alpha := \partial_\gamma \sigma(\gamma)|_{\gamma=\alpha}.$$

Then

$$\frac{\partial}{\partial\gamma}\big(- \log(\det(\Delta_{h_\gamma}))\big)\Big|_{\gamma=\alpha, \gamma\geq\alpha} = \mathrm{fp}_{t=0}\ \mathrm{Tr}_{L^2(M,g)}\Big(2\mathcal{M}_{\delta\sigma_\alpha}\big(e^{-t\Delta_g} - P_{\mathrm{Ker}(\Delta_g)}\big)\Big),$$

where $\mathcal{M}_f$ denotes the operator multiplication by the function f, $e^{-t\Delta_g}$ denotes the heat operator for (M, g) and $P_{\mathrm{Ker}(\Delta_g)}$ denotes the projection on the kernel of Δ_g.

This proposition is proved in §5. Generalizing our Polyakov formula to Euclidean polygons and surfaces with more than one conical singularity shall require additional considerations because one cannot change the angles independently. We expect that the results obtained here will help us to achieve these generalizations with the eventual goal of computing a closed formula for the determinant on planar sectors and Euclidean polygons. In the latter setting one naturally expects the following:

Conjecture 1. *Amongst all convex n-gons of fixed area, the regular one maximizes the determinant.*

We conclude this note by proving in section §6 the following proposition which shows that for the case of rectangular domains, the conjecture holds.

Proposition 2. *Let R be a rectangle of dimensions $L \times L^{-1}$. Then the zeta regularized determinant is uniquely maximized for $L = 1$, and tends to 0 as $L \to 0$ or equivalently as $L \to \infty$.*

Acknowledgements. *The authors are grateful to Gilles Carron and Rafe Mazzeo for their interest in the project and many useful conversations. The second author is also grateful to Lashi Bandara for productive discussions. We specially acknowledge Werner Müller's support and remarks.*

2. Geometric and analytic settings

2.1. **The determinant and Polyakov's formula.** Let us describe briefly the classical deduction of Polyakov's formula, since we will use the same argument. Let (M, g) be a smooth Riemannian surface with or without boundary. If $\partial M \neq \emptyset$, we consider the Dirichlet extension of the Laplacian, in which case $\mathrm{Ker}(\Delta_g) = \{0\}$. If $\partial M = \emptyset$, the Laplacian on M associated to the metric g is essentially self-adjoint and has non-trivial kernel. Since we want to simplify the exposition of the deduction of Polyakov's formula we keep all terms that may appear even if in some cases some terms may vanish.

Let $H_g(t, z, z')$ denote the heat kernel associated to Δ_g. It is the fundamental solution to the heat equation on M

$$\begin{aligned} (\Delta_g + \partial_t) H_g(t, z, z') &= 0 \quad (t > 0), \\ H_g(0, z, z') &= \delta(z - z'). \end{aligned}$$

The heat operator, $e^{-t\Delta_g}$ for $t > 0$, is trace class, and the trace is given by

$$\mathrm{Tr}(e^{-t\Delta_g}) = \int_M H_g(t, z, z) dz = \sum_{\lambda_k \geq 0} e^{-\lambda_k t}.$$

The zeta function and the heat trace are related by the Mellin transform

$$\zeta_{\Delta_g}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}(e^{-t\Delta_g} - P_{\mathrm{Ker}(\Delta_g)}) dt, \tag{2.1}$$

where $P_{\mathrm{Ker}(\Delta_g)}$ denotes the projection on the kernel of Δ_g.

It is well known that the heat trace has an asymptotic expansion for small values of t [12]. This expansion has the form

$$\mathrm{Tr}(e^{-t\Delta_g}) = a_0 t^{-1} + a_1 t^{\frac{-1}{2}} + a_2 + O(t^{\frac{1}{2}}).$$

The coefficients a_j are known as the heat invariants. They are given in terms of the curvature tensor and its derivatives as well as the geodesic curvature of the boundary in case of boundary. By (2.1) and the short time asymptotic expansion of the heat trace

$$\zeta_{\Delta_g}(s) = \frac{1}{\Gamma(s)} \left\{ \frac{a_0}{s-1} + \frac{a_1}{s - \frac{1}{2}} + \frac{a_2 - \dim(\mathrm{Ker}(\Delta_g))}{s} + e(s) \right\},$$

where $e(s)$ is an analytic function on $\mathrm{Re}(s) > 1$. This is how Ray and Singer proved that ζ_{Δ_g} is regular at $s = 0$, and the ζ regularized determinant of the Laplacian is indeed well-defined by (1.1).

Let $\{\sigma(\tau),\ \tau \in (-\epsilon, \epsilon)\}$ be a family of smooth conformal factors which depend on the parameter τ for some $\epsilon > 0$. Consider the corresponding family of conformal metrics $\{h_\tau = e^{2\sigma(\tau)} g,\ \tau \in (-\epsilon, \epsilon)\}$. To prove Polyakov's formula one first differentiates the spectral zeta function $\zeta_{\Delta_{h_\tau}}(s)$ with respect to τ. This requires differentiating the trace of the heat operator. Then, after integrating by parts, one obtains

$$\partial_\tau \zeta_{\Delta_{h_\tau}}(s) = \frac{s}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}\big(2\mathcal{M}_{\sigma'(\tau)}(e^{-t\Delta_{h_\tau}} - P_{\mathrm{Ker}(\Delta_{h_\tau})})\big)\, dt,$$

where $\mathcal{M}_{\sigma'(\tau)}$ denotes the operator multiplication by the function $\sigma'(\tau)$.

If the manifold is compact, and the metrics and the conformal factors are smooth, then the operator $\mathcal{M}_{\sigma'(\tau)}e^{-t\Delta_{h_\tau}}$ is trace class, and the trace behaves well for t large. As $t \to 0$ the trace also has an asymptotic expansion of the form

$$\begin{aligned}\mathrm{Tr}(\mathcal{M}_{\sigma'(\tau)}e^{-t\Delta_{h_\tau}}) \sim a_0(\sigma'(\tau), h_\tau)t^{-1} + a_1(\sigma'(\tau), h_\tau)t^{-\frac{1}{2}}\\ + a_2(\sigma'(\tau), h_\tau) - \dim(\mathrm{Ker}(\Delta_{h_\tau})) + O(t^{\frac{1}{2}})\end{aligned}$$

The notation $a_j(\sigma'(\tau), h_\tau)$ is meant to show that these are the coefficients of the given trace, which depend on $\sigma'(\tau)$ and on the metric h_τ. The dependence on the metric is through its associated heat operator.

Therefore, the derivative of $\zeta'_{\Delta_{h_\tau}}(0)$ at $\tau = 0$ is simply given by

$$\partial_\tau \zeta'_{\Delta_{h_\tau}}(0)\Big|_{\tau=0} = 2\ \big(a_2(\sigma'(0), h_0) - \dim(\mathrm{Ker}(\Delta_{h_0}))\big).$$

Polyakov's formula in (1.3) is exactly this equation.

2.2. **Euclidean sectors.** Let $S_\gamma \subset \mathbb{R}^2$ be a finite circular sector with opening angle $\gamma \in (0, \pi)$ and radius R. The Laplace operator Δ_γ with respect to the Euclidean metric is a priori defined on smooth functions with compact support within the open sector. It is well known that the Laplacian is not an essentially self adjoint operator since it has many self adjoint extensions; see e.g. [11] and [22]. The largest of these is the extension to

$$\mathrm{Dom}_{max}(\Delta_\gamma) = \{u \in L^2(S_\gamma) | \Delta_\gamma u \in L^2(S_\gamma)\}$$

For several reasons the most natural or standard self adjoint extension is the Friedrichs extension whose domain, $\mathrm{Dom}_F(\Delta_\gamma)$, is defined to be the completion of

$$C_0^\infty(S_\gamma) \text{ w.r.t the norm } \|\nabla f\|_{L^2}$$

intersected with Dom_{max}. For a smooth domain $\Omega \subset \mathbb{R}^2$, it is well known that

$$\mathrm{Dom}_F(\Delta_\Omega) = H_0^1(\Omega) \cap H^2(\Omega).$$

The same is true if the sector is convex which we shall assume; see [14, Theorem 2.2.3] and [20, Chapter 3, Lema 8.1].

Remark 1. *Let $S = S_{\gamma,R}$ be a planar circular sector of opening angle $\gamma \in (0, \pi)$, radius $R > 0$, and $S' = S_{\gamma',R'}$ be a circular sector of opening angle $\gamma' \in (0, \pi)$ and radius $R' > 0$. Then map $\Upsilon : S \to S'$ defined by $\Upsilon(\rho, \theta) = \left(\frac{R'\rho}{R}, \frac{\gamma'\theta}{\gamma}\right) = (r, \phi)$ induces a bijection*

$$\Upsilon^* : C_c^\infty(S') \xrightarrow{\cong} C_c^\infty(S), \quad f \mapsto \Upsilon^* f := f \circ \Upsilon.$$

This bijection extends to the domains of the Friedrichs extensions of the corresponding Laplace operator. Furthermore, under this map, the corresponding L^2 norms are equivalent, i.e., there exist constants $c, C > 0$ such that for any $f \in L^2(S')$,

$$c\|f\|_{L^2(S')} \leq \|\Upsilon^* f\|_{L^2(S)} \leq\ C\|f\|_{L^2(S')}.$$

The same holds for the norms on the corresponding Sobolev spaces H^k for $k \geq 0$. In spite of inducing an equivalence between the different domains, this map is not useful for our purposes since it does not produce a conformal transformation of the Euclidean metric.

To understand how the determinant of the Laplacian changes when the angle of the sector varies requires differentiating the spectral zeta function with respect to the angle

$$\frac{\partial}{\partial\gamma}\,\zeta_{S_\gamma}(s)=\frac{\partial}{\partial\gamma}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}_{L^2(S_\gamma,g)}(e^{-t\Delta_\gamma}-P_{\mathrm{Ker}(\Delta_\gamma)})dt. \tag{2.2}$$

In order to do that we use conformal transformations. Varying the sector is equivalent to varying a conformal family of metrics with singular conformal factors on a fixed domain. In this way, we can differentiate with respect to the opening angle by considering a fixed domain with a conformal family of metrics. It is natural to consider the fixed domain to be a sector as well.

2.2.1. *Conformal transformation from one sector to another.* Let (r,ϕ) denote polar coordinates on the sector S_γ. Since varying the radius is equivalent to scaling the sector by a constant factor, and the behavior of the Laplacian and its zeta-regularized determinant are well understood under scaling, we only need to understand the derivative with respect to the opening angle. Consequently, we shall assume the radii of all sectors considered from now on are fixed and equal to one.

The Euclidean metric on S_γ is

$$g=dr^2+r^2d\phi^2.$$

Let us consider another sector Q, with opening angle β, $Q:=S_\beta$, where β shall be chosen suitably: either $\beta:=\alpha$ or $\beta:=\alpha-\epsilon$ according to Lemma 3 where α is the angle at which we shall compute the derivative. The following considerations shall elucidate the subtleties involved in suitably choosing β as well as the necessity for two different values depending upon whether $\gamma\downarrow\alpha$ or $\gamma\uparrow\alpha$.

In order to avoid confusion we use (ρ,θ) to denote polar coordinates on Q. Then the Euclidean metric on Q in these coordinates is

$$g=d\rho^2+\rho^2d\theta^2.$$

Consider the map

$$\Psi_\gamma:Q\to S_\gamma,\quad (\rho,\theta)\mapsto\left(\rho^{\gamma/\beta},\frac{\gamma\theta}{\beta}\right)=(r,\phi) \tag{2.3}$$

The pull-back metric with respect to Ψ_γ of the Euclidean metric g on S_γ is

$$h_\gamma:=\Psi_\gamma^*g \;=\; \left(\frac{\gamma}{\beta}\right)^2\rho^{2\gamma/\beta-2}\left(d\rho^2+\rho^2d\theta^2\right)=e^{2\sigma_\gamma}\left(d\rho^2+\rho^2d\theta^2\right), \tag{2.4}$$

$$\sigma_\gamma(\rho,\theta) \;=\; \log\left(\frac{\gamma}{\beta}\rho^{\gamma/\beta-1}\right)=\log\left(\frac{\gamma}{\beta}\right)+\left(\frac{\gamma}{\beta}-1\right)\log\rho \tag{2.5}$$

Equation (2.4) shows that the pull-back metric h_γ is incomplete only when $0<\beta\le\gamma<\pi$. This is one of the reasons for the choice of β given in Lemma 3. We will consider the family of metrics

$$\{h_\gamma,\gamma\in[\beta,\pi)\}$$

defined by (2.4) on the fixed sector $Q=S_\beta$.

Remark 2. *The conformal factors* $\{\sigma_\gamma\}_{\gamma\in[\alpha-\epsilon,\alpha+\epsilon]}$ *in equation (2.5) can be considered as a family of the form* $\{\sigma_\tau(\rho,\theta)=\tau\ \xi(\rho,\theta)\}_{\tau\in[-\varepsilon,\varepsilon]}$. *Notice that*

$$\frac{\partial}{\partial\gamma}\sigma_\gamma\bigg|_{\gamma=\alpha}=\frac{1}{\alpha}+\frac{1}{\beta}\log(\rho)=:\xi(\rho,\theta),$$

therefore we can just take

$$\sigma_\tau(\rho,\theta)=\tau\left(\frac{1}{\alpha}+\frac{1}{\beta}\log(\rho)\right)$$

Even though there is no explicit expression relating the variables γ *and* τ*, this connection allows us to interpret the derivative with respect to the angle* γ *as the variation of the conformal factor in the direction of the function* $\xi(\rho,\theta)$ *which is singular as* $\rho\downarrow 0$.

The area element on Q with respect to the metric h_γ is

$$dA_{h_\gamma}=e^{2\sigma_\gamma}\rho d\rho d\theta=e^{2\sigma_\gamma}dA_g, \tag{2.6}$$

and the Laplace operator Δ_{h_γ} associated to the metric h_γ is formally given by

$$\Delta_{h_\gamma}=-\left(\frac{\beta}{\gamma}\right)^2\rho^{-2\gamma/\beta+2}\left(\partial_\rho^2+\rho^{-1}\partial_\rho+\rho^{-2}\partial_\theta^2\right)=e^{-2\sigma_\gamma}\Delta, \tag{2.7}$$

where $\Delta:=\Delta_\beta=-\partial_\rho^2-\rho^{-1}\partial_\rho-\rho^{-2}\partial_\theta^2$ is the Laplacian on (Q,g).

The transformation Ψ_γ induces a map between the function spaces

$$\Psi_\gamma^*:C_c^\infty(S_\gamma)\to C_c^\infty(Q),\quad f\mapsto\Psi_\gamma^*f:=f\circ\Psi_\gamma.$$

However, since the transformation of the metric is not of class C^1 up to the boundary for γ close enough to β, it is not clear a priori that the domains of the Laplacians are transformed in the same way as in the smooth case. In spite of that, the following Proposition shows that the correspondence is preserved.

Proposition 3. *For* $\gamma\geq\beta$*, the map* Ψ_γ^* *gives an equivalence between the domain of* Δ_{h_γ} *and the domain of the Dirichlet self-adjoint extension of* Δ_γ *on the sector* S_γ. *Moreover,*

$$\Psi_\gamma^*(\mathrm{Dom}(\Delta_\gamma))=\mathrm{Dom}(\Delta_{h_\gamma})=H^2(Q,h_\gamma)\cap H_0^1(Q,h_\gamma),$$

with $\Delta_{h_\gamma}=e^{-2\sigma_\gamma}\Delta_\beta$.

This proposition is a direct consequence of the following two Lemmas.

Lemma 1. *The map* Ψ_γ *defined by equation (2.3) induces an isometry* Ψ_γ^* *between the Sobolev spaces* $H_0^1(Q,h_\gamma)$ *and* $H_0^1(S_\gamma,g_\gamma)$ *for* $\gamma\geq\beta$.

Proof. As before, let r,ϕ denote the coordinates in S_γ, and let ρ,θ denote the coordinates in Q. The volume element in Q and the Laplacian for the metric h_γ are given in (2.6) and (2.7), respectively.

The transformation Ψ_γ^* extends to the L^2 spaces. The fact that Ψ_γ^* is an isometry between $L^2(S_\gamma,g)$ and $L^2(Q,h_\gamma)$ follows from a standard change of variables computation. For $f:S_\gamma\to\mathbb{R}$,

$$\int_{S_\gamma}|f(r,\phi)|^2rdrd\phi=\int_Q|f\circ\Psi_\gamma|^2\left(\frac{\gamma}{\beta}\right)^2\ \rho^{2\frac{\gamma}{\beta}-1}d\rho d\theta=\int_Q|\Psi_\gamma^*f|^2e^{2\sigma_\gamma}\rho d\rho d\theta.$$

Then it is clear that if $\gamma \geq \beta$, $2\gamma/\beta - 1 \geq 1$, so the L^2-spaces are equivalent, and the L^2 norms $L^2(S_\gamma, g)$ and $L^2(Q, h_\gamma)$ coincide.

Next let $f \in H_0^1(S_\gamma, g)$. To prove that $\Psi_\gamma^* f \in H_0^1(Q, h_\gamma)$ we show that the L^2-norms $\|df\|_{L^2(S_\gamma,g)}$ and $\|df \circ d\Psi_\gamma\|_{L^2(Q,h_\gamma)}$ are equivalent. Since $|df|_g^2 = |\nabla_g f|^2 = g^{lj}(\partial_l f)(\partial_j f)$,

$$\int_{S_\gamma} |\nabla_g f|^2 dA_g = \int_Q \left(\left(\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \phi}\right)^2 \right) \circ \Psi_\gamma(\rho,\theta) \right) e^{2\sigma_\gamma} \rho d\rho d\theta.$$

Using $\Psi_\gamma^* f = f \circ \Psi_\gamma(\rho,\theta)$ we have

$$\frac{\partial f}{\partial r}(\Psi_\gamma(\rho,\theta)) = \frac{\beta}{\gamma}\rho^{1-\gamma/\beta}\frac{\partial \Psi_\gamma^* f}{\partial \rho}, \quad \frac{\partial f}{\partial \phi}(\Psi_\gamma(\rho,\theta)) = \frac{\beta}{\gamma}\frac{\partial \Psi_\gamma^* f}{\partial \theta}.$$

Substituting above, we obtain

$$\begin{aligned}
\int_{S_\gamma} |\nabla_g f|^2 dA_g &= \int_Q \left(\left(\frac{\beta}{\gamma}\rho^{1-\gamma/\beta}\frac{\partial \Psi_\gamma^* f}{\partial \rho}\right)^2 + \rho^{-2\gamma/\beta}\left(\frac{\beta}{\gamma}\frac{\partial \Psi_\gamma^* f}{\partial \theta}\right)^2 \right) e^{2\sigma_\gamma}\rho d\rho d\theta \\
&= \int_Q \left(\frac{\beta}{\gamma}\rho^{1-\gamma/\beta}\right)^2 \left(\left(\frac{\partial \Psi_\gamma^* f}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\beta}{\gamma}\frac{\partial \Psi_\gamma^* f}{\partial \theta}\right)^2 \right) e^{2\sigma_\gamma}\rho d\rho d\theta \\
&= \int_Q e^{-2\sigma_\gamma} \left(\left(\frac{\partial \Psi_\gamma^* f}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial \Psi_\gamma^* f}{\partial \theta}\right)^2 \right) e^{2\sigma_\gamma}\rho d\rho d\theta \\
&= \int_Q |\nabla_{h_\gamma} \Psi^* f|^2 dA_{h_\gamma}.
\end{aligned}$$

This completes the proof. □

Lemma 2. *The map Ψ_γ^* provides an isometry between the Sobolev spaces $H^2(Q, h_\gamma)$ and $H^2(S_\gamma, g)$, for $\gamma \geq \beta$. A function $f \in H^2(Q, h_\gamma)$ if and only if $\Psi^* f \in H^2(S_\gamma, g)$.*

Proof. Let $f \in H^2(Q, h_\gamma)$. By definition $\Psi_\gamma^* f = (f \circ \Psi_\gamma)(\rho,\theta)$, so

$$|\Delta_{h_\gamma} \Psi_\gamma^* f|^2 = \left(\frac{\beta}{\gamma}\right)^2 \rho^{-\frac{4\gamma}{\beta}+4} \left(\frac{\partial^2 \Psi_\gamma^* f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Psi_\gamma^* f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Psi_\gamma^* f}{\partial \theta^2} \right)^2.$$

Since

$$\frac{\partial^2 \Psi_\gamma^* f}{\partial \rho^2} = \left(\frac{\gamma}{\beta}\right)^2 \rho^{2\frac{\gamma}{\beta}-2}\frac{\partial^2 f}{\partial r^2}(\Psi_\gamma(\rho,\theta)) + \frac{\gamma}{\beta}\left(\frac{\gamma}{\beta}-1\right)\rho^{\frac{\gamma}{\beta}-2}\frac{\partial f}{\partial r}(\Psi_\gamma(\rho,\theta)),$$

it is easy to see that

$$\int_Q |\Delta_{h_\gamma}\Psi_\gamma^* f|^2 dA_{h_\gamma} = \int_Q (|\Delta_g f|^2 \circ \Psi_\gamma)(\rho,\theta) e^{2\sigma_\gamma} dA_g = \int_{S_\gamma} |\Delta_g f|^2 dA_g$$

where the last equality follows from the standard change of variables, and g denotes the Euclidean metric on both Q and S_γ. □

Example 1. *Let $\gamma \in [\beta, \pi)$, and h_γ be as above. Let $\varphi(\rho,\theta) := \rho^x \sin(k\pi\theta/\beta)$. It is easy to see that*

- $\varphi \in L^2(Q, h_\gamma) \Leftrightarrow x > -\gamma/\beta$
- $\varphi \in H^1(Q, h_\gamma) \Leftrightarrow x > 0$

- $\varphi \in H^2(Q, h_\gamma) \Leftrightarrow x > \frac{\gamma}{\beta}$.

The example above shows that the domain of the Laplacian depends on the angle, and in particular, it will be different for different angles. As a consequence several problems appear here that distinguish this case from the classical smooth case and force us to go into the details of the differentiation process.

2.2.2. *Domains of the Laplace operators.* Even though the description of the domains of the family of Laplace operators $\{\Delta_{h_\gamma}, \gamma \geq \beta\}$ given in the previous section is useful for our purposes, it is not enough. Unlike the smooth case, this family do not act on a single fixed Hilbert space when γ varies but instead we will demonstrate below that they act on a nested family of weighted, so-called "b"-Sobolev spaces.

Definition 1. *The b-vector fields on (S_γ, g), denoted by $\mathcal{V}_b$, are the $\mathcal{C}^\infty$ span of the vector fields*

$$\mathcal{V}_b := \mathcal{C}^\infty \ \text{span of } \{r\partial_r, \partial_\phi\},$$

where $\mathcal{C}^\infty$ means that the coefficient functions are smooth up to the boundary. For $m \in \mathbb{N}$, the b-Sobolev space is defined as

$$H_b^m := \{f \mid V_1 \ldots V_j f \in L^2(S_\gamma, g)\, \forall j \leq m,\, V_1, \ldots, V_j \in \mathcal{V}_b\},$$

and $H_b^0 = L^2(S, g)$. The weighted b-Sobolev spaces are

$$r^x H_b^m = \{f \mid \exists v \in H_b^m, \quad f = r^x v\}.$$

We first apply results due to several authors, including but not limited to, Mazzeo [25] Theorem 7.14 and Lesch [22] Proposition 1.3.11.

Proposition 4. *The domain of the Laplace operator Δ_γ on the sector S_γ with Dirichlet boundary condition is*

$$\mathrm{Dom}(\Delta_\gamma) = r^2 H_b^2 \cap H_0^1(S_\gamma, g).$$

Proof. By equation (19) in [26] and Theorem 7.14 [25] (c.f. [22] Proposition 1.3.11), any element in the domain of the Friedrichs extension of Laplacian Δ_γ has a partial expansion near $r = 0$ of the form

$$\sum_{\gamma_j \in]-n/2, -n/2+2]} c_j r^{\gamma_j} \psi_j(\phi) + w, \quad w \in r^2 H_b^2.$$

In our case the dimension $n = 2$, and the indicial roots γ_j are given by

$$\gamma_j = \pm\sqrt{\mu_j},$$

where μ_j is an eigenvalue of the Laplacian on the link of the singularity, and ψ_j is the eigenfunction with eigenvalue μ_j. The link is in this case $[0, \gamma]$ with Dirichlet boundary condition. These eigenvalues are therefore $\mu_j = \frac{j^2\pi^2}{\gamma^2}$ with $j \in \mathbb{N}$, $j \geq 1$. In particular, there are *no* indicial roots in the critical interval $]-1, 1]$, because $\gamma < \pi$. Taking into account the Dirichlet boundary condition away from the singularity, it follows that the domain of the Laplace operator is precisely given by

$$r^2 H_b^2(S_\gamma) \cap H_0^1(S_\gamma, g).$$

□

The operators Δ_{h_γ}, albeit each defined on functions on Q, have domains which are defined in terms of $L^2(Q, dA_{h_\gamma})$. In particular, the area forms depend on γ. Consequently, in order to fix a single Hilbert space on which our operators act, we use the following maps

$$\begin{aligned} \Phi_\gamma &: L^2(Q, dA_{h_\gamma}) \to L^2(Q, dA), \quad f \mapsto e^{\sigma_\gamma} f = \frac{\gamma}{\beta}\rho^{\gamma/\beta - 1} f; \\ \Phi_\gamma^{-1} &: L^2(Q, dA) \to L^2(Q, dA_{h_\gamma}), \quad f \mapsto e^{-\sigma_\gamma} f = \frac{\beta}{\gamma}\rho^{-\gamma/\beta+1} f. \end{aligned} \tag{2.8}$$

Each Φ_γ is an isometry of $L^2(Q, dA_{h_\gamma})$ and $L^2(Q, dA)$, since

$$\int_Q f^2 dA_{h_\gamma} = \int_Q f^2 e^{2\sigma_\gamma} dA = \int_Q (\Phi_\gamma f)^2 dA.$$

Proposition 5. *For all $\gamma \in [\beta, \pi)$, we have*

$$\Phi_\gamma\left(\mathrm{Dom}(\Delta_{h_\gamma})\right) \subseteq \rho^{2\gamma/\beta} H_b^2(Q, dA) \cap H_0^1(Q, dA).$$

Moreover,

$$\Phi_\gamma\left(\mathrm{Dom}(\Delta_{h_\gamma})\right) \subset \Phi_{\gamma'}\Big(\mathrm{Dom}(\Delta_{h_{\gamma'}})\Big), \quad \gamma' < \gamma.$$

Proof. Let us start by comparing the H_b^2 spaces. We first compute that

$$r = \rho^{\gamma/\beta} \implies \rho\partial_\rho = \frac{\gamma}{\beta} r\partial_r,$$

and

$$\partial_\theta = \frac{\gamma}{\beta}\partial_\phi.$$

Recall that $\mathrm{Dom}(\Delta_{h_\gamma}) = \Psi_\gamma^*(\mathrm{Dom}(\Delta_\gamma))$. In the same way as in the proof of Proposition 3, we obtain

$$\begin{aligned} \Psi_\gamma^*(H_b^2(S_\gamma)) &= H_b^2(Q, dA_{h_\gamma}) \\ &= \{f \mid f, Vf, V_1V_2 f \in L^2(Q, dA_{h_\gamma}),\ \forall V, V_1, V_2 \in \mathcal{C}^\infty\langle \rho\partial\rho, \partial_\theta\rangle\}. \end{aligned}$$

Let $f \in r^2 H_b^2(S_\gamma)$, by definition $f(r, \phi) = r^2 u(r, \phi)$ with $u \in H_b^2(S_\gamma)$. Then

$$(\Psi_\gamma^* f)(\rho, \theta) = f(\Psi_\gamma(\rho, \theta)) = f(\rho^{\gamma/\beta}, \gamma\theta/\beta) = \rho^{2\gamma/\beta}(\Psi_\gamma^* u)(\rho, \theta)$$

It therefore follows that

$$(\Psi_\gamma^*(r^2 H_b^2(S_\gamma)) = \rho^{2\gamma/\beta} H_b^2(Q, dA_{h_\gamma}),$$

Notice that

$$\begin{aligned} H_b^2(Q, dA_{h_\gamma}) &= \rho^{-\gamma/\beta+1} H_b^2(Q, dA) \\ \Psi_\gamma^*(r^2 H_b^2(S_\gamma)) &= \rho^{\gamma/\beta+1} H_b^2(Q, dA), \end{aligned}$$

and

$$\begin{aligned} \Phi_\gamma(\Psi_\gamma^*(r^2 H_b^2(S_\gamma)) &= \Phi_\gamma(\rho^{\gamma/\beta+1} H_b^2(Q, dA)) \\ &= \rho^{2\gamma/\beta} H_b^2(Q, dA) \subseteq \rho^2 H_b^2(Q, dA), \end{aligned}$$

for $\gamma \in [\beta, \pi)$.

Finally, we see that

$$\gamma' < \gamma \implies \rho^{2\gamma/\beta} H_b^2(Q, dA) \subset \rho^{2\gamma'/\beta} H_b^2(Q, dA).$$

Now, we claim that

$$\Phi_\gamma\left(H_0^1(Q,dA_{h_\gamma})\cap\rho^{2\gamma/\beta}H_b^2(Q,dA_{h_\gamma})\right)\subseteq H_0^1(Q,dA).$$

Note that $\mathcal{C}_0^\infty(Q)$ is independent of h_γ. Then, it is enough to show that for any $f\in\mathrm{Dom}(\Delta_{h_\gamma})$ the $L^2(Q,dA)$-norms of $\Phi_\gamma f$ and $\nabla(\Phi_\gamma f)$, can be estimated using the fact that $f\in H_0^1(Q,dA_{h_\gamma})\cap\rho^2H_b^2(Q,dA)$. By definition, Φ_γ is an isometry of $L^2(Q,dA_{h_\gamma})$ and $L^2(Q,dA)$. So we only need to prove that $\nabla(\Phi_\gamma f)\in L^2(Q,dA)$.

We know that

$$\int_Q|\nabla_{h_\gamma}f|^2dA_{h_\gamma}=\int_Q e^{-2\sigma_\gamma}\left((\partial_\rho f)^2+\rho^{-2}(\partial_\theta f)^2\right)e^{2\sigma_\gamma}dA=\int_Q|\nabla f|^2dA.$$

Next we compute

$$\begin{aligned}\int_Q|\nabla\Phi_\gamma f|^2dA&=\int_Q\left((\partial_\rho e^{\sigma_\gamma}f)^2+\rho^{-2}(\partial_\theta e^{\sigma_\gamma}f)^2\right)dA\\&=\int_Q\left\{e^{2\sigma_\gamma}\left((\partial_\rho f)^2+\rho^{-2}(\partial_\theta f)^2\right)+(\partial_\rho e^{\sigma_\gamma})^2f^2+2(\partial_\rho e^{\sigma_\gamma})e^{\sigma_\gamma}f(\partial_\rho f)\right\}dA\end{aligned}$$

The first term

$$\int_Q e^{2\sigma_\gamma}\left((\partial_\rho f)^2+\rho^{-2}(\partial_\theta f)^2\right)dA=\int_Q|\nabla f|^2\rho^{2\frac{\gamma}{\beta}-2}\frac{\gamma^2}{\beta^2}dA\le\frac{\gamma^2}{\beta^2}\int_Q|\nabla_{h_\gamma}f|^2dA_{h_\gamma}$$

since $\frac{\gamma}{\beta}\ge1$, $\rho^{2\frac{\gamma}{\beta}-2}\le1$ on Q.

To estimate the second term, we use that $f\in\rho^{\gamma/\beta+1}H_b^2(Q,dA)$, therefore

$$\int_Q(\partial_\rho e^{\sigma_\gamma})^2f^2dA=c\int_Q f^2\rho^{2\frac{\gamma}{\beta}-4}dA\le\int_Q f^2\rho^{-\frac{\gamma}{\beta}-1}dA<\infty,$$

where $c=\frac{\gamma^2}{\beta^2}\frac{(\gamma-\beta)^2}{\beta^2}$ and we have used again that $\gamma\ge\beta$. For the third term we compute

$$\begin{aligned}\int_Q(\partial_\rho e^{\sigma_\gamma})e^{\sigma_\gamma}f(\partial_\rho f)dA&=c\int_Q\rho^{2\frac{\gamma}{\beta}-3}f(\partial_\rho f)dA\\&\le c\left(\int_Q f^2\rho^{2\frac{\gamma}{\beta}-4}dA\right)^{1/2}\left(\int_Q(\rho\partial_\rho f)^2\rho^{2\frac{\gamma}{\beta}-4}dA\right)^{1/2}<\infty\end{aligned}$$

Putting everything together, we have proven that

$$\Phi_\gamma(\Psi_\gamma^*(\mathrm{Dom}(\Delta_\gamma)))\subseteq\rho^{2\gamma/\beta}H_b^2(Q,dA)\cap H_0^1(Q,dA).$$

In order to see that for $\beta\le\gamma'<\gamma<\pi$,

$$\Phi_\gamma(\Psi_\gamma^*(\mathrm{Dom}(\Delta_\gamma)))\subset\Phi_{\gamma'}(\Psi_{\gamma'}^*(\mathrm{Dom}(\Delta_{\gamma'})),$$

we first note that

$$\Phi_\gamma(\Psi_\gamma^*(\mathrm{Dom}(\Delta_\gamma)))\subset\rho^{2\gamma/\beta}H_b^2(Q,dA)\subset\rho^{2\gamma'/\beta}H_b^2(Q,dA).$$

Next, we must show that

$$f\in H_0^1(Q,dA_{h_\gamma})\implies\Phi_{\gamma'}^{-1}\Phi_\gamma f\in H_0^1(Q,dA_{h_{\gamma'}}),\quad\gamma'<\gamma.$$

The L^2 norm of $\nabla_{h_{\gamma'}}(\Phi_{\gamma'}^{-1}\Phi_\gamma f)$ can be estimated in the same way as above using the fact that $\gamma'<\gamma$ so that $\gamma-\gamma'>0$. □

2.2.3. *The family of operators.* Finally, let us introduce the family of operators that we will use to prove Polyakov's formula. Let us define H_γ as

$$H_\gamma := \Phi_\gamma \circ \Psi_\gamma \circ \Delta_\gamma \circ \Psi_\gamma^{-1} \circ \Phi_\gamma^{-1} = \Phi_\gamma \circ \Delta_{h_\gamma} \circ \Phi_\gamma^{-1}. \tag{2.9}$$

The domains of the family $\{H_\gamma\}_\gamma$ nest

$$\beta \le \gamma' \le \gamma \implies \mathrm{Dom}(H_\gamma) \subset \mathrm{Dom}(H_{\gamma'}) \subset \mathrm{Dom}(\Delta)$$

where Δ is the Laplacian on Q.

3. Proof of the variational Polyakov formula

Let A be an integral operator on $L^2(Q, h_\gamma)$ with kernel $K_A(z, z')$. The transformed operator $\Phi_\gamma A \Phi_\gamma^{-1}$ to the Hilbert space $L^2(Q, g)$ by the conformal transformation $\Phi_\gamma f = e^{\sigma_\gamma} f$ has integral kernel $e^{\sigma_\gamma(z)} K_A(z, z') e^{\sigma_\gamma(z')}$. This follows from the transformation of the area element and

$$\begin{aligned}
(\Phi_\gamma A \Phi_\gamma^{-1} f)(z) &= \Phi_\gamma \left(\int_Q K_A(z, z') e^{-\sigma_\gamma(z')} f(z') dA_{h_\gamma}(z') \right) \\
&= e^{\sigma_\gamma(z)} \int_Q K_A(z, z') e^{-\sigma_\gamma(z')} f(z') e^{2\sigma_\gamma(z')} dA \\
&= \int_Q e^{\sigma_\gamma(z)} K_A(z, z') e^{\sigma_\gamma(z')} f(z') dA(z')
\end{aligned}$$

for $f \in L^2(Q, g)$.

Thus

$$\begin{aligned}
\mathrm{Tr}_{L^2(Q,g)} \left(\Phi_\gamma A \Phi_\gamma^{-1} \right) &= \int_Q K_A(z, z) e^{2\sigma_\gamma(z)} dA(z) \\
&= \int_Q K_A(z, z) dA_{h_\gamma}(z) = \mathrm{Tr}_{L^2(Q, h_\gamma)} (A).
\end{aligned}$$

3.1. Differentiation of the operators.

As we saw in equation (2.9), the domains of the family $\{H_\gamma\}_\gamma$ nest. In order to compute the derivative with respect to the angle at $\gamma = \alpha$, one would like to apply both H_γ and H_α to the elements in the domain of H_α. There are subtleties which arise, but we can remedy them.

Lemma 3. *Let $0 < \alpha < \pi$, $0 < \epsilon$, $\beta \le \alpha$ and $\beta \le \gamma < \pi$. Then the following one-sided derivatives*

$$\left.\frac{dH_\gamma}{d\gamma^-}\right|_{\gamma=\alpha} \text{ with } \beta = \alpha - \epsilon, \quad \text{ and } \quad \left.\frac{dH_\gamma}{d\gamma^+}\right|_{\gamma=\alpha} \text{ with } \beta = \alpha$$

are well defined. In both cases we have

$$\dot{H}_\gamma = \frac{\partial H_\gamma}{\partial \gamma} = \left(\frac{\partial \sigma_\gamma}{\partial \gamma} \right) H_\gamma + \Phi_\gamma \left(\frac{\partial \Delta_{h_\gamma}}{\partial \gamma} \right) \Phi_\gamma^{-1} - \Phi_\gamma \Delta_{h_\gamma} \left(\frac{\partial \sigma_\gamma}{\partial \gamma} \right) \Phi_\gamma^{-1}. \tag{3.1}$$

Proof. The formal expression for $\dot{H}_\gamma$ follows from a straightforward computation. For the left derivative, $\gamma < \alpha$. As noticed in §2 if $\gamma < \beta$ in equation (2.4), the metric h_γ is complete. Consequently we define $\beta := \alpha - \epsilon$. Since $\mathrm{Dom}(H_\alpha) \subset \mathrm{Dom}(H_\gamma)$ for each $\gamma < \alpha$, we can apply both the operators H_α and H_γ to all elements of the domain of H_α and let $\gamma \uparrow \alpha$. The derivative $\left.\frac{dH_\gamma}{d\gamma^-}\right|_{\gamma=\alpha}$ is therefore computed in this way and given by (3.1). We can then let $\epsilon \to 0$.

For the right derivative $\gamma > \alpha$, we let $\beta := \alpha$, and Q is the sector at which we differentiate. In this case we cannot apply both operators H_γ and H_α to all elements of $\mathrm{Dom}(H_\alpha)$ because there might be functions $f \in \mathrm{Dom}(H_\alpha) \setminus \mathrm{Dom}(H_\gamma)$. However, for such a function there is a sequence $\{f_n\}_n$ in $C_0^\infty(Q, g)$ with $f_n \to f$ in $\mathrm{Dom}(H_\alpha)$, since smooth and compactly supported functions are dense in the domain of the operator. Then, for $f \in \mathrm{Dom}(H_\alpha) \setminus \mathrm{Dom}(H_\gamma)$ we define

$$\frac{dH_\gamma}{d\gamma^+}\bigg|_{\gamma=\alpha} f = \lim_{n\to\infty} \lim_{\gamma \downarrow \alpha} \frac{H_\alpha f_n - H_\gamma f_n}{\alpha - \gamma}, \tag{3.2}$$

and we shall see that this limit is well defined.

Taking $\gamma = \alpha = \beta$ in (3.1) and $f \in C_0^\infty$,

$$\begin{aligned}
\frac{dH_\gamma}{d\gamma^+}\bigg|_{\gamma=\alpha} f &= \frac{1}{\alpha}\left(1 + \log(\rho)\right)\Delta_\alpha f + -2\frac{1}{\alpha}\left(1 + \log(\rho)\right)\Delta_\alpha f - \Delta_\alpha\left(\frac{1}{\alpha}\left(1 + \log(\rho)\right) f\right) \\
&= -\frac{1}{\alpha}\left(1 + \log(\rho)\right)\Delta_\alpha f - \Big(\big(\frac{1}{\alpha}\left(1 + \log(\rho)\right)\Delta_\alpha f \\
&\quad -2g(\nabla_\alpha(\frac{1}{\alpha}\left(1 + \log(\rho)\right), \nabla_\alpha f) + \frac{1}{\alpha}\mathcal{M}_{\Delta_\alpha(1+\log(\rho))} f\Big) \\
&= -\frac{2}{\alpha}\left(1 + \log(\rho)\right)\Delta_\alpha f + \frac{2}{\alpha}\rho^{-1}\partial_\rho f
\end{aligned}$$

Now, we show that the limit (3.2) exists and is unique. For $f \in \mathrm{Dom}(H_\alpha) = \mathrm{Dom}(\Delta_\alpha)$ with $\{f_n\}_n$ be a sequence in C_0^∞ such that $f_n \to f$ in $\mathrm{Dom}(H_\alpha)$, then

$$\Delta_\alpha f_n \to \Delta_\alpha f, \quad \rho^{-1}\partial_\rho f_n \to \rho^{-1}\partial_\rho f, \text{ in } L^2(Q, g).$$

By the Cauchy-Schwarz inequality, we have convergence

$$(\log\rho)\Delta_\alpha f_n \to (\log\rho)\Delta_\alpha f,$$

in $L^1(Q, g)$. Consequently,

$$\frac{dH_\gamma}{d\gamma^+}\bigg|_{\gamma=\alpha} f_n \to \frac{dH_\gamma}{d\gamma^+}\bigg|_{\gamma=\alpha} f.$$

Since the right side is well-defined for all $f \in \mathrm{Dom}(H_\alpha)$ and independent of the choice of sequence $f_n \in C_0^\infty$, the limit (3.2) is well defined. $\square$

Remark 3. *Although the definitions of σ_γ, h_γ, Q, and H_γ depend on the choice of β, the final variational formula is independent of this choice since, in the end, everything is pulled back to the original sector S_α, and β drops out of the equations. We only require this parameter to rigorously differentiate the trace; the sector $Q = S_\beta$ and the choice of β are part of an auxiliary construction.*

Proposition 6. *Let H_γ be as in equation (2.9). Then the derivative of the transformed heat operators is*

$$\begin{aligned}
\frac{d}{d\gamma}\mathrm{Tr}_{L^2(Q,g)}(\Phi_\gamma e^{-t\Delta_{h_\gamma}}\Phi_\gamma^{-1}) &= -t\ \mathrm{Tr}_{L^2(Q,g)}(\dot{H}_\gamma e^{-tH_\gamma}) \\
&= -t\ \mathrm{Tr}_{L^2(Q,h_\gamma)}(\dot{\Delta}_{h_\gamma} e^{-t\Delta_{h_\gamma}}),
\end{aligned}$$

where $\dot{\Delta}_{h_\gamma} \equiv \frac{\partial}{\partial\gamma}\ \Delta_{h_\gamma}\Big|_\gamma = -2(\partial_\gamma\sigma_\gamma)\Delta_{h_\gamma}$.

Proof. Although the proof of this proposition is standard in the boundaryless case, we include some details to show that the statement also holds in our case. Following the same computation as in [3, Lemma 5.1] and [29],

$$\frac{d}{d\gamma}\mathrm{Tr}_{L^2(Q,g)}(\Phi_\gamma e^{-t\Delta_{h_\gamma}}\Phi_\gamma^{-1}) = \mathrm{Tr}_{L^2(Q,g)}\left(\frac{d}{d\gamma}e^{-tH_\gamma}\right).$$

Let $\gamma_2 > \gamma_1$. Duhamel's principle is well known and often used in the settings of both manifolds with boundaries and conical singularities; see [8]. We apply this principle in terms of the operators

$$e^{-tH_{\gamma_1}} - e^{-tH_{\gamma_2}} = \int_0^t -e^{-sH_{\gamma_1}}H_{\gamma_1}e^{-(t-s)H_{\gamma_2}} + e^{-sH_{\gamma_1}}H_{\gamma_2}e^{-(t-s)H_{\gamma_2}}\ ds.$$

Notice that the product $H_{\gamma_1}e^{-(t-s)H_{\gamma_2}}$ is well defined since $e^{-(t-s)H_{\gamma_2}}$ maps $L^2(Q,g)$ onto $\mathrm{Dom}(H_{\gamma_2})$ and $\mathrm{Dom}(H_{\gamma_2}) \subset \mathrm{Dom}(H_{\gamma_1})$. Then for $f \in L^2(Q,g)$, $e^{-(t-s)H_{\gamma_2}}f \in \mathrm{Dom}(H_{\gamma_1})$.

Dividing by $\gamma_1 - \gamma_2$ the previous equation and letting $\gamma_2 \to \gamma_1$, we obtain

$$\frac{d}{d\gamma}\ e^{-tH_\gamma}\Big|_{\gamma=\gamma_1} = -\int_0^t e^{-sH_{\gamma_1}}\left(\frac{d}{d\gamma}H_\gamma\Big|_{\gamma=\gamma_1}\right)e^{-(t-s)H_{\gamma_1}}\ ds.$$

Therefore since the heat operators are trace class

$$\frac{d}{d\gamma}\mathrm{Tr}_{L^2(Q,g)}(\Phi_\gamma e^{-t\Delta_{h_\gamma}}\Phi_\gamma^{-1}) = -t\ \mathrm{Tr}_{L^2(Q,g)}\left(\dot{H}_\gamma e^{-tH_\gamma}\right). \tag{3.3}$$

We computed $\frac{\partial}{\partial\gamma}H_\gamma$ in equation (3.1). Substituting its value into our calculation above we obtain

$$\begin{aligned}
&\mathrm{Tr}_{L^2(Q,g)}\left(\dot{H}_\gamma e^{-tH_\gamma}\right)\\
&= \mathrm{Tr}_{L^2(Q,g)}\left(((\partial_\gamma\sigma_\gamma)H_\gamma + \Phi_\gamma\left(\partial_\gamma\Delta_{h_\gamma}\right)\Phi_\gamma^{-1} - \Phi_\gamma\Delta_{h_\gamma}\left(\partial_\gamma\sigma_\gamma\right)\Phi_\gamma^{-1})e^{-tH_\gamma}\right)\\
&= \mathrm{Tr}_{L^2(Q,g)}\left(\Phi_\gamma\left((\partial_\gamma\sigma_\gamma)\Delta_{h_\gamma}e^{-t\Delta_{h_\gamma}} + (\partial_\gamma\Delta_{h_\gamma})e^{-t\Delta_{h_\gamma}} - \Delta_{h_\gamma}(\partial_\gamma\sigma_\gamma)e^{-t\Delta_{h_\gamma}}\right)\Phi_\gamma^{-1}\right)\\
&= \mathrm{Tr}_{L^2(Q,h_\gamma)}\left((\partial_\gamma\sigma_\gamma)\Delta_{h_\gamma}e^{-t\Delta_{h_\gamma}} + \dot{\Delta}_{h_\gamma}e^{-t\Delta_{h_\gamma}} - \Delta_{h_\gamma}(\partial_\gamma\sigma_\gamma)e^{-t\Delta_{h_\gamma}}\right)\\
&= \mathrm{Tr}_{L^2(Q,h_\gamma)}\left(\dot{\Delta}_{h_\gamma}e^{-t\Delta_{h_\gamma}}\right),
\end{aligned}$$

provided that $(\partial_\gamma\sigma_\gamma)H_\gamma e^{-tH_\gamma}$, $\Phi_\gamma\left(\partial_\gamma\Delta_{h_\gamma}\right)\Phi_\gamma^{-1}e^{-tH_\gamma}$, and $\Phi_\gamma\Delta_{h_\gamma}\left(\partial_\gamma\sigma_\gamma\right)\Phi_\gamma^{-1}e^{-tH_\gamma}$ are trace class in $L^2(Q,g)$. We show in Lemma 4 that this is the case. $\square$

Proof of Theorem 1. In order to prove Theorem 1, we differentiate the spectral zeta function with respect to the angle γ as in equation (2.2).

From Proposition 6 we have

$$\begin{aligned}
\frac{\partial}{\partial\gamma}\ \mathrm{Tr}_{L^2(S_\gamma,g)}(e^{-t\Delta_\gamma})\Big|_{\gamma=\alpha} &= \frac{\partial}{\partial\gamma}\ \mathrm{Tr}_{L^2(Q,h_\gamma)}(e^{-t\Delta_{h_\gamma}})\Big|_{\gamma=\alpha}\\
= \frac{\partial}{\partial\gamma}\ \mathrm{Tr}_{L^2(Q,g)}(\Phi_\gamma e^{-t\Delta_{h_\gamma}}\Phi_\gamma^{-1})\Big|_{\gamma=\alpha} &= \frac{\partial}{\partial\gamma}\ \mathrm{Tr}_{L^2(Q,g)}(e^{-tH_\gamma})\Big|_{\gamma=\alpha}\\
&= -t\ \mathrm{Tr}_{L^2(Q,h_\alpha)}\left(\dot{\Delta}_{h_\alpha}e^{-t\Delta_{h_\alpha}}\right)
\end{aligned}$$

where as before $H_\gamma = \Phi_\gamma \Delta_{h_\gamma} \Phi_\gamma^{-1}$. In this way we obtain

$$\frac{\partial}{\partial \gamma} \operatorname{Tr}_{L^2(S_\gamma,g)}(e^{-t\Delta_\gamma})\Big|_{\gamma=\alpha} = 2t \operatorname{Tr}_{L^2(Q,h_\alpha)}\left(\left(\frac{1}{\alpha}+\frac{1}{\beta}\log\rho\right)\Delta_{h_\alpha} e^{-t\Delta_{h_\alpha}}\right),$$

where we have replaced $(\delta\sigma_\alpha)$ by its value $\left(\frac{1}{\alpha}+\frac{1}{\beta}\log\rho\right)$, and we have used that the Laplacian changes conformally in dimension 2. On the other hand,

$$\frac{\partial}{\partial t}\operatorname{Tr}_{L^2(Q,h_\alpha)}\left((\delta\sigma_\alpha)e^{-t\Delta_{h_\alpha}}\right) = -\operatorname{Tr}_{L^2(Q,h_\alpha)}\left((\delta\sigma_\alpha)\Delta_{h_\alpha} e^{-t\Delta_{h_\alpha}}\right).$$

Thus

$$\frac{\partial}{\partial \gamma} \operatorname{Tr}_{L^2(S_\gamma,g)}(e^{-t\Delta_\gamma})\Big|_{\gamma=\alpha} = -2t \frac{\partial}{\partial t}\operatorname{Tr}_{L^2(Q,h_\alpha)}\Big((\delta\sigma_\alpha)e^{-t\Delta_{h_\alpha}}\Big).$$

In order to compute $\delta\zeta'_{\Delta_\alpha}(0)$ we consider

$$\frac{\partial}{\partial \gamma}\zeta_{\Delta_\gamma}(s)\Big|_{\gamma=\alpha} = -\frac{2}{\Gamma(s)}\int_0^\infty t^s \frac{\partial}{\partial t}\operatorname{Tr}_{L^2(Q,h_\alpha)}\Big((\delta\sigma_\alpha)e^{-t\Delta_{h_\alpha}}\Big)dt.$$

Here we use integration by parts. In order to be able to integrate by parts, we require appropriate estimates of the trace for large and small values of t. The large values of t are not problematic since

$$\operatorname{Tr}_{L^2(Q,h_\alpha)}\Big((\delta\sigma_\alpha)e^{-t\Delta_{h_\alpha}}\Big) = O(e^{-c'_\alpha t}), \text{ as } t\to\infty,$$

for some constant $c'_\alpha > 0$. This statement follows from a standard argument; see for example [3, Lemma 5.2]. Let $t > 1$ and write

$$(\delta\sigma_\alpha)e^{-t\Delta_{h_\alpha}} = (\delta\sigma_\alpha)e^{-\frac{1}{2}\Delta_{h_\alpha}} e^{-(t-\frac{1}{2})\Delta_{h_\alpha}}.$$

The operator $(\delta\sigma_\alpha)e^{-\frac{1}{2}\Delta_{h_\alpha}}$ is trace class. Since the spectrum of the operator Δ_{h_α} is contained in $[c_\alpha,\infty)$ for some $c_\alpha > 0$, for $t > 1$ we have

$$\|e^{-(t-\frac{1}{2})\Delta_{h_\alpha}}\|_{L^2(Q,h_\alpha)} \le e^{-c_\alpha(t-\frac{1}{2})}$$

Thus for any $t > 0$, the trace satisfies the desired estimate:

$$\begin{aligned}|\operatorname{Tr}((\delta\sigma_\alpha)e^{-t\Delta_{h_\alpha}}| &\le \|(\delta\sigma_\alpha)e^{-\frac{1}{2}\Delta_{h_\alpha}} e^{-(t-\frac{1}{2})\Delta_{h_\alpha}}\|_1 \\ &\le \|(\delta\sigma_\alpha)e^{-\frac{1}{2}\Delta_{h_\alpha}}\|_1 \|e^{-(t-\frac{1}{2})\Delta_{h_\alpha}}\|_{L^2(Q,h_\alpha)} \ll e^{-c'_\alpha t},\end{aligned}$$

where $\|\cdot\|_1$ denotes the trace norm of the operator and $\|\cdot\|_{L^2(Q,h_\alpha)}$ denotes the operator norm in $L^2(Q,h_\alpha)$. For small values of t, the existence of an asymptotic expansion of the trace follows from the existence of the corresponding expansion of the heat kernel.

After doing integration by parts we have

$$\frac{\partial}{\partial_\gamma}\zeta_{\Delta_\gamma}(s)\Big|_{\gamma=\alpha} = \frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{Tr}_{L^2(Q,h_\alpha)}\Big(2(\delta\sigma_\alpha)e^{-t\Delta_{h_\alpha}}\Big)dt,$$

Therefore, the standard argument now shows that the variation of $\zeta'_{\Delta_\alpha}(0) = -\log(\det(\Delta_\alpha))$ in the angular direction is given by the constant term in the asymptotic expansion for small time of

$$\operatorname{Tr}_{L^2(Q,h_\alpha)}\left(2\left(\frac{1}{\alpha}+\frac{1}{\beta}\log\rho\right)e^{-t\Delta_{h_\alpha}}\right).$$

Recalling the change of variables (2.3) we go back to the original sector S_α, and obtain that this is the constant term in the asymptotic expansion of

$$\operatorname{Tr}_{L^2(S_\alpha,h_\alpha)}\left(\frac{2}{\alpha}\left(1+\log(r)\right)e^{-t\Delta_\alpha}\right) = \int_{S_\alpha}\frac{2}{\alpha}\left(1+\log(r)\right)H_{S_\alpha}(t,r,\phi,r,\phi)r\,dr\,d\phi \tag{3.4}$$

as $t\to 0$. $\square$

3.2. **Heat kernel estimates.** The last ingredients in the proof of the variational formula are estimates on the heat kernel which show that the operators listed at the end of the proof of Proposition 6 are trace class. We do not need a sharp estimate, a general estimate in terms of the time variable is enough for our purposes. We obtain such an estimate from [10] and [1].

Proposition 7. *Let S denote a finite Euclidean sector or a surface with an isolated conical singularity. Then the heat kernel of the Friedrichs extension of Laplacian on S satisfies the following estimates*

$$\begin{aligned} |H(t,z,z')| &\le \frac{C}{t}, \\ |\partial_t H(t,z,z')| &\le \frac{C}{t^2}, \end{aligned}$$

for all $z,z'\in S$, and $t\in(0,T)$, where $C>0$ is a fixed constant which depends only on the constant $T>0$.

Proof. Sectors and surfaces with an isolated conical singularity are both rather mild examples of stratified spaces. Consequently, the heat kernel satisfies the estimate (2.1) on p. 1062 of [1]. This estimate is

$$H(t,z,z')\le Ct^{-1},\quad \forall z,z'\in S,\quad \forall t\in(0,1), \tag{3.5}$$

since the dimension $n=2$.

Next, we apply the results by E.B. Davies in [10] which hold for the Laplacian on a general Riemannian manifold whose balls are compact if the radius is sufficiently small. These minimal hypotheses are satisfied for both geometric settings considered here. By [10, Lemma 1],

$$|H(t,z,z')|^2\le H(t,z,z)H(t,z',z'),$$

for all $z,z'\in S$, and all $t>0$. If $T<1$, then this estimate together with (3.5) gives the first estimate in the Proposition. In general, by [10] the function $t\mapsto H(t,z,z)$ is positive, monotone decreasing in t, and log convex for every z. For a fixed $T\ge 1$, the estimate (3.5) together with the above shows that

$$|H(t,z,z')|^2\le C^2\quad \forall t\ge 1.$$

So, we simply replace the constant C with the constant CT, which we again denote by C and obtain the estimate

$$|H(t,z,z')|^2\le C^2t^{-2},\quad \forall t\in(0,T),\quad \forall z \text{ and } z'\in S.$$

Next, we apply Theorem 3 of [10], which states that the time derivatives of the heat kernel satisfy the estimates

$$\left|\frac{\partial^n}{\partial t^n}H(t,z,z')\right|\le\frac{n!}{(t-s)^n}H(s,z,z)^{1/2}H(s,z',z')^{1/2},\quad n\in\mathbb{N},\quad 0<s<t.$$

Making the special choice $s = t/2$ and $n = 1$, we have

$$|\partial_t H(t,z,z')| \leq \frac{2}{t} H(t/2,z,z)^{1/2} H(t/2,z',z')^{1/2}.$$

Using the estimates for the heat kernel we estimate the right side above which shows that

$$|\partial_t H(t,z,z')| \leq Ct^{-2}, \quad \forall t \in (0,T), \quad \forall z,z' \in S.$$

□

Remark 4. *By the heat equation, the estimate for the time derivative of the heat kernel implies the following estimate for the Laplacian of the heat kernel*

$$|\Delta H(t,z,z')| \leq Ct^{-2},$$

for any $0 < t < T$, and $z, z' \in S$, for a constant $C > 0$ depending on T.

Let us go back to the trace class property of the operators in question. An easy way to see that an operator is trace class is to write it either as a product of a bounded operator and a trace class operator, or as the product of two Hilbert-Schmidt operators. Since in all cases $\Delta e^{-t\Delta}$ and $e^{-t\Delta}\Delta$ are trace class, it is enough to prove that $\mathcal{M}_\xi \Delta e^{-t\Delta}$ is Hilbert Schmidt, where $\mathcal{M}_\xi$ denotes the operator multiplication by a function $\xi(z)$ that behaves as $\log(\rho)$ in a neighborhood of the singular point $\rho = 0$.

Lemma 4. *Let S denote a finite sector or a surface with an isolated conical singularity. Assume that the angle of the sector or the angle at the conical singularity in the surface is $\alpha \in (0,\pi)$. Let $\mathcal{M}_\psi$ denote the operator multiplication by a function ψ. Let ξ be a continuous function on $S \setminus \{\rho = 0\}$ that behaves as $\log(\rho)$ in a neighborhood of the singular point $\rho = 0$. Then, for any $t > 0$ the following operators*

(1) $\mathcal{M}_\xi e^{-t\Delta}$
(2) $\mathcal{M}_\xi \Delta e^{-t\Delta}$
(3) $\mathcal{M}_\psi e^{-t\Delta}$, *where $\psi(\rho,\theta) = O(\rho^{-c})$ as $\rho \to 0$, for $c < 1$.*

are Hilbert-Schmidt.

Proof. Recall that an integral operator is Hilbert-Schmidt if the L^2-norm of its integral kernel is finite. For simplicity, let us write the proof for the case of a finite sector S with opening angle α and radius R. Using the estimates given in Proposition 7 we have that

$$\begin{aligned}\|\mathcal{M}_\xi e^{-t\Delta}\|_{HS} &\leq C \int_{S\times S} |\log(\rho)|^2 |H(t,z,z')|^2 dA dA' \\ &\leq \widetilde{C}(\alpha,R,t) \int_0^R \int_0^R |\log(\rho)|^2 \rho\rho' d\rho d\rho' < \infty,\end{aligned}$$

since $|\log(\rho)|^2\rho$ is bounded on $(0,R)$. Thus $\mathcal{M}_\xi \Delta e^{-t\Delta}$ is a Hilbert-Schmidt operator.

Similarly,

$$\begin{aligned}\|\mathcal{M}_\psi e^{-t\Delta}\|_2 &\leq C \int_{S\times S} |\psi(z)|^2 |H(t,z,z')|^2 dA dA' \\ &\leq \widetilde{C}(\alpha,R,t) \int_0^R \int_0^R \rho^{-2c+1} \rho' d\rho d\rho' < \infty\end{aligned}$$

since $c < 1$. The same arguments together with the compactness of the surface away from the singularity imply the result for surfaces. $\square$

4. The quarter circle

We have proven that the derivative of the logarithm of the determinant of the Laplacian in the angular direction on a finite Euclidean sector is given in terms of the constant term in the small time expansion in (1.4) in Theorem 1. However we still need to determine the existence of such an expansion and the finiteness of the coefficient.

4.1. **General method.** To compute the asymptotic expansion of the trace on the right hand side of equation (1.4), we replace the heat kernel by a parametrix. One may partition the domain and use suitable parametrices for each part and combine these using cut-off functions to make a parametrix for the whole domain. For finite sectors we use the following parametrices for each corresponding part of the domain.

(1) The heat kernel for $\mathbb{R}^2$ for the interior away from the straight edges.
(2) The heat kernel for $\mathbb{R}^2_+$ for neighborhoods of the straight edges away from the corners.
(3) The heat kernel for the unit disk for a neighborhood of the curved arc away from the corners.
(4) The heat kernel for the infinite sector with opening angle $\pi/2$ for neighborhoods of each of the corners of the circular arc which meet the straight edges.
(5) The heat kernel for the infinite sector with opening angle α for a neighborhood of the vertex of the sector with opening angle α.

The salient point which is well known to experts, is that this patchwork parametrix restricted to the diagonal is asymptotically equal to the true heat kernel on the diagonal with an error of $O(t^\infty)$ as $t \downarrow 0$ (c.f. Lemma 2.2 of [27]). Consequently, it suffices to compute the contributions to the integral in (1.4) from each parametrix integrating locally using cut-off functions. Similar to the short time asymptotic expansion of the heat trace which has an extra purely local contribution from the angles, it is natural to expect that the angles also appear in the variational formula for the determinant.

4.2. **Proof of Theorem 2.** Let $\alpha = \pi/2$, then the infinite sector with angle α is the quadrant $C = \{(x, y) \in \mathbb{R}^2, x, y \geq 0\}$. The Dirichlet heat kernel in this case can be obtained as the product of the Dirichlet heat kernel on the half line $[0, \infty)$ with itself. For $x_1, x_2 \in [0, \infty)$ the Dirichlet heat kernel is given by

$$p_{hl}(t, x_1, x_2) = \frac{1}{\sqrt{4\pi t}}(e^{-\frac{(x_1-x_2)^2}{4t}} - e^{-\frac{(x_1+x_2)^2}{4t}}).$$

Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ be in C, we have

$$\begin{aligned} p_C(t, u, v) &= p_{hl}(t, x_1, x_2)p_{hl}(t, y_1, y_2) \\ &= \frac{1}{4\pi t}(e^{-\frac{|u-v|^2}{4t}} + e^{-\frac{|u+v|^2}{4t}} - e^{-\frac{(x_1-x_2)^2+(y_1+y_2)^2}{4t}} - e^{-\frac{(x_1+x_2)^2+(y_1-y_2)^2}{4t}}). \end{aligned}$$

Writing this in polar coordinates with $u = re^{i\phi}$, $v = r'e^{i\phi'}$ we obtain

$$\begin{aligned} p_C(t,u,v) &= \frac{e^{-\frac{r^2+r'^2}{4t}}}{4\pi t}(e^{\frac{rr'}{2t}\cos(\phi'-\phi)} + e^{-\frac{rr'}{2t}\cos(\phi'-\phi)} \\ &\quad -e^{\frac{rr'}{2t}\cos(\phi'+\phi)} - e^{-\frac{rr'}{2t}\cos(\phi'+\phi)}) \\ &= \frac{e^{-\frac{r^2+r'^2}{4t}}}{2\pi t}\left(\cosh\left(\frac{rr'\cos(\phi'-\phi)}{2t}\right) - \cosh\left(\frac{rr'\cos(\phi'+\phi)}{2t}\right)\right). \end{aligned}$$

As we have explained in §4.1, in order to find the contribution to the constant term in the expansion of (1.4) for small t which comes from integrating near the corner at $r = 0$, we use a partition of unity to approximate the heat kernel in the finite sector close to the corner by the heat kernel of the infinite sector. We then integrate along the diagonal in a neighborhood of the corner. We shall see below that this contribution to the constant term is independent of the size of this neighborhood of the corner and is therefore well defined. Let $R > 0$, then the contribution of the corner is given then by the finite part at $t = 0$ of the following integral

$$\begin{aligned} I &= \int_0^R \int_0^{\pi/2} \frac{4}{\pi}(1+\log(r))p_C(t,r,\phi,r,\phi)\ r\ dr\ d\phi \\ &= \int_0^R \int_0^{\pi/2} \frac{4}{\pi}(1+\log(r))\frac{e^{-\frac{r^2}{2t}}}{4\pi t}(e^{\frac{r^2}{2t}} + e^{-\frac{r^2}{2t}} - e^{\frac{r^2}{2t}\cos(2\phi)} - e^{-\frac{r^2}{2t}\cos(2\phi)})\ rd\phi dr \\ &= \frac{1}{\pi^2 t}\int_0^R \int_0^{\pi/2} (1+\log(r))(1+e^{-\frac{r^2}{t}} - e^{-\frac{r^2}{2t}}e^{\frac{r^2}{2t}\cos(2\phi)} - e^{-\frac{r^2}{2t}}e^{-\frac{r^2}{2t}\cos(2\phi)})\ rd\phi dr. \end{aligned}$$

Recall that the factor $\frac{4}{\pi}(1+\log(r))$ is in this case the contribution due to the conformal factor $\frac{2}{\alpha}(1+\log r)$ since $\alpha = \frac{\pi}{2}$.

We split this integral into two different terms T_1 and T_2, the first of which

$$\begin{aligned} T_1(t) &= \frac{1}{\pi^2 t}\int_0^R \int_0^{\pi/2} (1+\log(r))(1+e^{-\frac{r^2}{t}})\ rd\phi dr \\ &= \frac{1}{2\pi t}\int_0^R (1+\log(r))(1+e^{-\frac{r^2}{t}})\ rdr \\ &= \frac{1}{2\pi t}\left(\int_0^R rdr + \int_0^R \log(r)\ rdr + \int_0^R e^{-\frac{r^2}{t}}\ rdr + \int_0^R \log(r)e^{-\frac{r^2}{t}}\ rdr\right). \end{aligned}$$

The first two terms clearly do not contribute to the t^0 coefficient, so we discard them and look only at the constant term in the expansion in t of

$$\widetilde{T}_1(t) = \frac{1}{2\pi t}\left(\int_0^R e^{-\frac{r^2}{t}}\ rdr + \int_0^R \log(r)e^{-\frac{r^2}{t}}\ rdr\right). \tag{4.1}$$

We compute

$$\frac{1}{2\pi t}\int_0^R e^{-\frac{r^2}{t}}\ rdr = \frac{1}{4\pi}\int_0^{R^2/t} e^{-u}\ du = \frac{1}{4\pi} - \frac{1}{4\pi}e^{-R^2/t},$$

and

$$\frac{1}{2\pi t}\int_0^R \log(r)e^{-\frac{r^2}{t}}\ rdr = \frac{1}{4\pi}\int_0^{R^2/t}\log(u)e^{-u}\ du$$
$$= \frac{1}{4\pi}\int_0^\infty \log(u)e^{-u}\ du - \frac{1}{4\pi}\int_{R^2/t}^\infty \log(u)e^{-u}\ du = \frac{-\gamma_e}{4\pi} - \frac{1}{4\pi}\int_{R^2/t}^\infty \log(u)e^{-u}\ du$$

where γ_e is the Euler constant. Using integration by parts, we obtain for the second integral

$$\begin{aligned}\int_{R^2/t}^\infty \log(u)e^{-u}\ du &= \log(R^2/t)e^{-R^2/t} + \int_{R^2/t}^\infty \frac{e^{-u}}{u}\ du \\ &= \log(R^2/t)e^{-R^2/t} + E_1(R^2/t).\end{aligned}$$

Then for $\widetilde{T}_1(t)$ we obtain

$$\widetilde{T}_1(t) = \frac{1}{4\pi} - \frac{1}{4\pi}e^{-R^2/t} - \frac{\gamma_e}{4\pi} - \frac{\log(R^2/t)}{4\pi}e^{-R^2/t} - \frac{1}{4\pi}E_1(R^2/t),$$

where

$$E_1(x) := \int_x^\infty \frac{e^{-s}}{s}ds.$$

The second and fourth terms in this expression for $\widetilde{T}_1(t)$ decay rapidly as $t \downarrow 0$ and so do not contribute to the t^0 coefficient in the asymptotic expansion. We show here that the last term decays similarly by approximating the function $xe^xE_1(x)$ as in [24, p201]

$$\frac{x}{x+1} < xe^x\int_x^\infty s^{-1}e^{-s}\ ds < \frac{x+1}{x+2}.$$

Then for $x = R^2/t$, and $t \leq 1$ we obtain

$$\frac{t}{R^2+t}e^{-R^2/t} < E_1(R^2/t) < \frac{t}{R^2}e^{-R^2/t}\frac{R^2+t}{R^2+2t}$$

which shows that the last term in the expression for $\widetilde{T}_1(t)$ also decays rapidly as $t \downarrow 0$. Therefore the contribution of $T_1(t)$ to the constant term is given by

$$\mathrm{fp}_{t=0}\ T_1(t) = \frac{1}{4\pi} - \frac{\gamma_e}{4\pi}. \tag{4.2}$$

Let us consider now the second term $T_2(t)$

$$\begin{aligned}T_2(t) &= -\frac{1}{\pi^2 t}\int_0^R\int_0^{\pi/2}(1+\log(r))(e^{-\frac{r^2}{2t}}e^{\frac{r^2}{2t}\cos(2\phi)} + e^{-\frac{r^2}{2t}}e^{-\frac{r^2}{2t}\cos(2\phi)})\ rd\phi dr \\ &= -\frac{1}{\pi^2 t}\int_0^R(1+\log(r))e^{-\frac{r^2}{2t}}\int_0^{\pi/2}(e^{\frac{r^2}{2t}\cos(2\phi)} + e^{-\frac{r^2}{2t}\cos(2\phi)})\ d\phi\ rdr.\end{aligned}$$

The modified Bessel function of first kind of order zero admits the following integral representation

$$I_0(a) = \frac{1}{\pi}\int_0^\pi e^{a\cos(\phi)}\ d\phi$$

for $a \in \mathbb{R}$, $a \geq 0$. After a change of variables

$$\int_0^{\pi/2} e^{a\cos(2\phi)}\ d\phi = \frac{\pi}{2} I_0(a).$$

Since $\cos(\pi - x) = -\cos(x)$, we obtain

$$T_2(t) = -\frac{1}{\pi t}\int_0^R (1+\log(r)) e^{-\frac{r^2}{2t}} I_0\left(\frac{r^2}{2t}\right)\ rdr.$$

We know how to compute these integrals using techniques inspired by [36]. Let us write $T_{2,1}$ for the integral with 1, and $T_{2,2}$ for the integral with $\log(r)$, so $T_2 = T_{2,1} + T_{2,2}$. We start by changing variables $u = r^2/2t$

$$T_{2,1} = -\frac{1}{\pi t}\int_0^R r e^{-r^2/2t} I_0\left(\frac{r^2}{2t}\right) dr = -\frac{1}{\pi}\int_0^{R^2/2t} e^{-u} I_0(u) du.$$

Let $I_1(u)$ be the modified Bessel function of first kind of order one. By [37, (3) p. 79] with $\nu = 1$,

$$uI_1'(u) + I_1(u) = uI_0(u). \tag{4.3}$$

By [37, (4) p. 79] with $\nu = 0$,

$$uI_0'(u) = uI_1(u). \tag{4.4}$$

We use these to calculate

$$\begin{aligned}
\frac{d}{du}\left(e^{-u}u(I_0(u)+I_1(u))\right) &= e^{-u}\left(-uI_0(u) - uI_1(u) + I_0(u) + I_1(u) + uI_0'(u) + uI_1'(u)\right) \\
&= e^{-u}\left(-uI_1(u) + I_0(u) + uI_0'(u)\right), \qquad \text{by (4.3)} \\
&= e^{-u} I_0(u), \qquad \text{by (4.4)}.
\end{aligned}$$

Next, define

$$g(u) := e^{-u}u(I_0(u) + I_1(u)), \tag{4.5}$$

and note that we have computed

$$g'(u) = e^{-u} I_0(u).$$

We therefore have

$$-\frac{1}{\pi}\int_0^{R^2/2t} e^{-u} I_0(u) du = -\frac{1}{\pi}\left(g(R^2/2t) - g(0)\right).$$

The Bessel functions are known to satisfy (see [37])

$$I_0(0) = 1, \quad I_1(0) = 0.$$

It follows that $g(0) = 0$, and we therefore obatain that

$$-\frac{1}{\pi}\int_0^{R^2/2t} e^{-u} I_0(u) du = -\frac{1}{\pi} g(R^2/2t).$$

For large arguments, the Bessel functions admit the following asymptotic expansions (see [37])

$$I_j(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{1}{2x}\left(j^2 - \frac{1}{4}\right) + \sum_{k=2}^{\infty} c_{j,k} x^{-k}\right), \quad x \gg 0, \quad j = 0, 1.$$

We therefore compute the expansion of g as follows

$$g(u) = \frac{\sqrt{u}}{\sqrt{2\pi}} \left(2 - \frac{1}{4u} + \sum_{k=2}^{\infty} (c_{0,k} + c_{1,k}) u^{-k} \right), \quad u \gg 1.$$

Consequently, for $u = R^2/2t$ we have

$$g(R^2/2t) = \frac{R}{\sqrt{4\pi t}} \left(2 - \frac{t}{2R^2} + \sum_{k=2}^{\infty} (c_{0,k} + c_{1,k}) \left(\frac{2t}{R^2} \right)^k \right), \quad t \ll 1.$$

It follows that for small t, $T_{2,1}(t)$ has the following asymptotic expansion

$$T_{2,1}(t) = -\frac{R}{\pi\sqrt{4\pi t}} \left(2 - \frac{t}{2R^2} + \sum_{k=2}^{\infty} (c_{0,k} + c_{1,k}) \left(\frac{2t}{R^2} \right)^k \right), \quad t \ll 1.$$

Therefore $T_{2,1}$ does not contribute to constant term.

Now, let us look at $T_{2,2}$. Changing variables again $u = r^2/2t$ we obtain

$$\begin{aligned} T_{2,2} &= -\frac{1}{\pi t} \int_0^R r \log(r) e^{-r^2/2t} I_0 \left(\frac{r^2}{2t} \right) dr = -\frac{1}{\pi} \int_0^{R^2/2t} \log(\sqrt{2tu}) e^{-u} I_0(u) du \\ &= -\frac{1}{2\pi} \int_0^{R^2/2t} \log(u) e^{-u} I_0(u) du - \frac{\log(2t)}{2\pi} \int_0^{R^2/2t} e^{-u} I_0(u) du. \end{aligned}$$

For the first integral we use (4.5) and integrate by parts,

$$\int_0^{R^2/2t} \log(u) e^{-u} I_0(u) du = \log(u) g(u)|_0^{R^2/2t} - \int_0^{R^2/2t} e^{-u}(I_0(u) + I_1(u)) du.$$

Since $g'(u) = e^{-u} I_0(u)$, we have

$$\int_0^{R^2/2t} e^{-u}(I_0(u) + I_1(u)) du = g(R^2/2t) - g(0) + \int_0^{R^2/2t} e^{-u} I_1(u) du.$$

Note that $I_0'(u) = I_1(u)$. Therefore, we integrate by parts again,

$$\begin{aligned} \int_0^{R^2/2t} e^{-u} I_1(u) du &= e^{-u} I_0(u)|_0^{R^2/2t} - \int_0^{R^2/2t} -e^{-u} I_0(u) du, \\ &= e^{-u} I_0(u)|_0^{R^2/2t} + g(u)|_0^{R^2/2t}. \end{aligned}$$

Putting these calculations together, we have

$$\int_0^{R^2/2t} \log(u) e^{-u} I_0(u) du = \log(u) g(u)|_0^{R^2/2t} - 2 \left(g(R^2/2t) - g(0) \right) - e^{-u} I_0(u)|_0^{R^2/2t}.$$

Therefore, we have calculated

$$\begin{aligned} -\frac{1}{2\pi} \int_0^{R^2/2t} \log(u) e^{-u} I_0(u) du &= \frac{1}{2\pi} \Big(-\log(u) g(u)|_0^{R^2/2t} + \\ &\qquad 2 \left(g(R^2/2t) - g(0) \right) + e^{-u} I_0(u)|_0^{R^2/2t} \Big); \\ -\frac{\log(2t)}{2\pi} \int_0^{R^2/2t} e^{-u} I_0(u) du &= -\frac{\log(2t)}{2\pi} \left(g(R^2/2t) - g(0) \right). \end{aligned}$$

Since $g(0)=0$ and $I_0(0)=1$, we have

$$\begin{aligned}
&T_{2,2}(t)\\
&= \frac{1}{2\pi}\Big(-\log(R^2/2t)g(R^2/2t)+2g(R^2/2t)+e^{-R^2/2t}I_0(R^2/2t)-1-\log(2t)g(R^2/2t)\Big)\\
&= \frac{1}{2\pi}\Big(-2\log(R)g(R^2/2t)+2g(R^2/2t)+e^{-R^2/2t}I_0(R^2/2t)-1\Big).
\end{aligned}$$

We use the asymptotic expansion of $I_0(u)$ for $u\to\infty$ to compute

$$e^{-R^2/2t}I_0(R^2/2t)=\frac{\sqrt{t}}{R\sqrt{\pi}}\left(1+\frac{t}{4R^2}+\sum_{k=2}^{\infty}c_{0,k}\left(\frac{2t}{R^2}\right)^k\right),\quad t\ll 1.$$

We therefore obtain that the asymptotic expansion of $T_{2,2}(t)$ is

$$\begin{aligned}
&-\frac{1}{2\pi}-\frac{R\log R}{\pi\sqrt{4\pi t}}\left(2-\frac{t}{2R^2}+\sum_{k=2}^{\infty}(c_{0,k}+c_{1,k})\left(\frac{2t}{R^2}\right)^k\right)\\
&+\frac{R}{\pi\sqrt{4\pi t}}\left(2-\frac{t}{2R^2}+\sum_{k=2}^{\infty}(c_{0,k}+c_{1,k})\left(\frac{2t}{R^2}\right)^k\right)\\
&+\frac{\sqrt{t}}{2\pi R\sqrt{\pi}}\left(1+\frac{t}{4R^2}+\sum_{k=2}^{\infty}c_{0,k}\left(\frac{2t}{R^2}\right)^k\right),\quad t\ll 1.
\end{aligned}$$

Therefore the contribution of $T_{2,2}$ is $-\frac{1}{2\pi}$. Putting the contributions of T_1 and T_2 together, we obtain that the contribution of the corner with angle $\pi/2$ to the constant term in the asymptotic expansion of the trace in (1.4) is

$$\mathrm{fp}_{t=0}\int_0^R\int_0^{\pi/2}\left(\frac{4}{\pi}(1+\log(r))\right)p_C(t,r,\phi,r,\phi)\ r\ dr\ d\phi = -\frac{1}{4\pi}-\frac{\gamma_e}{4\pi}. \tag{4.6}$$

Finally we use the suitable parametrices given at the beginning of §4.1 together with Polyakov's formula in the smoothly bounded case for the interior and the boundary away from the corners to compute the finite part in the integral in equation (3.4):

$$\begin{aligned}
&\mathrm{fp}_{t=0}\int_{S_{\pi/2}}\frac{4}{\pi}\left(1+\log(r)\right)H_{S_\alpha}(t,r,\phi,r,\phi)rdrd\phi\\
&=\int_{S_{\pi/2}}\frac{4}{\pi}\left(1+\log(r)\right)\frac{1}{24\pi}\mathrm{Scal}_g rdrd\phi+\int_{\partial S_{\pi/2}}\frac{4}{\pi}\left(1+\log(r)\right)\frac{1}{12\pi}\kappa_g ds\\
&\quad +3\ \mathrm{fp}_{t=0}\int_0^{\epsilon}\int_0^{\pi/2}\left(\frac{4}{\pi}(1+\log(r))\right)p_C(t,r,\phi,r,\phi)\ r\ dr\ d\phi\\
&=\frac{1}{6\pi}-\frac{3}{4\pi}-\frac{3\gamma_e}{4\pi}\sim -0.3234\ldots
\end{aligned}$$

□

5. A surface with an isolated conical singularity

In this section we consider a Riemannian surface with one or more isolated conical singularities and conformal transformations of the metric that represent a change in the cone angle at only one conical singularity while the cone angles of the other conical singularities stay constant. Thus, we assume that the conformal factors are smooth up to all singularities except one. Because of that, and for the simplicity in the arguments to follow, no generality shall be lost if we assume that there is only one conical singularity.

Let (M, g) be a Riemannian surface with a conical singularity at a point $p \in M$ with opening angle γ. Then p has a neighborhood $p \in \mathcal{N} \subset M$ such that

$$\mathcal{N} \cong [0,1]_r \times S^1_\phi,$$

and the Riemannian metric restricted to this neighborhood is given by

$$g|_{\mathcal{N}} = dr^2 + r^2\gamma^2 d\phi^2 =: g_\gamma|_{\mathcal{N}},$$

where $d\phi^2$ is the standard metric on S^1 with radius one. The conical singularity p is defined by $r = 0$. In order to keep track of the cone angle we denote (M, g) as (M, g_γ) or simply M_γ.

Proposition 8. *The domain of the Friedrichs extension of the Laplace operator on a surface (M, g_γ) with an isolated conical singularity with opening angle γ and radial coordinate r near the singularity is*

$$\mathrm{Dom}(\Delta_{M_\gamma}) = \mathbb{R} + r^2 H^2_b(M, g_\gamma) = \{u \mid \exists u_0 \in \mathbb{R},\, v \in r^2 H^2_b(M, g_\gamma),\, u = u_0 + v\}.$$

Proof. Following the argument in the proof of Proposition 4, in this case there is precisely one indicial root in the critical interval, namely 0. The corresponding eigenfunction is the constant function, and since there is no other boundary, the domain of the operator is completely characterized as

$$u : \exists u_0 \in \mathbb{R}, \quad v \in r^2 H^2_b, \quad u = u_0 + v.$$

□

Analogous to the case of finite sectors treated in section 2.2.1, if we consider a finite cone given by $(\mathcal{N}, g_\gamma)$ with $\mathcal{N} = [0,1]_r \times S^1_\phi = [0,1]_\rho \times S^1_\theta$ and the map

$$\Psi_\gamma : \mathcal{N} \to \mathcal{N}, \quad (\rho, \theta) \mapsto \left(\rho^{\gamma/\beta}, \theta\right) = (r, \phi).$$

The pull back of the metric g_γ is a metric on $\mathcal{N}$ conformal to g_β

$$\Psi^*_\gamma g_\gamma = e^{2\sigma_\gamma}\left(d\rho^2 + \rho^2\beta^2 d\theta^2\right),$$

where σ_γ is the same function as in the case of the sector; see equation (2.5).

In a finite exact cone, as well as in the case of sectors, the map Ψ_γ is globally defined, and it provides a nice geometric interpretation of the conformal factors. On a general surface the map Ψ_γ is only defined near the singularity, not globally. Even though it could be extended smoothly to $M \setminus \{p\}$, there is no guarantee that the resulting pull-back metric will be conformal to g_β. That is why it is more convenient to work directly with a conformal family of metrics on (M, g) with suitable restrictions on the conformal factors. Furthermore, for surfaces with conical singularities, we set $\alpha = \beta$, so all the metrics are conformal to g_α. We are now ready to define the family of conformal metrics for which Proposition 1 is valid.

Definition 2. *Let (M, g_α) be a fixed surface with an isolated conical singularity at $p \in M$ with opening angle α. Let $\{\sigma_\gamma, \gamma \in [\alpha, \pi)\}$ be a family of functions on M, satisfying the following conditions:*

(1) $\sigma_\gamma \in C^\infty(M \setminus \{p\})$;
(2) σ_γ *depends smoothly on* γ;

(3) *on $\mathcal{N}$, σ_γ is given by equation (2.5) with $\beta = \alpha$,*

$$\sigma_\gamma|_{\mathcal{N}}(\rho,\theta) = \log\left(\frac{\gamma}{\alpha}\right) + \left(\frac{\gamma}{\alpha} - 1\right)\log\rho. \tag{5.1}$$

We define the family of conformal of metrics as

$$\{h_\gamma = e^{2\sigma_\gamma} g_\alpha, \gamma \in [\alpha, \pi)\}$$

The Friedrichs Laplacian on M with respect to the metric h_γ is denoted by Δ_{h_γ}.

We now require a description of the domains of the operators Δ_{h_γ} and the relationship between these as γ varies. Most of the computations for the sector imply the analogous results in this case even though the map Ψ_γ is defined only locally near the conical singularity, where the conformal factor has the precise form given by (5.1). This is due to the smoothness of the conformal factor away from the singularity and the compactness of M. Therefore the same proof of Lemmas 1 and 2 give an equivalence of the Sobolev spaces for the metric h_γ and a metric g_γ on $\mathcal{N}$. In addition, the same argument as in the first part of the proof of Proposition 5 shows that

$$\mathrm{Dom}(\Delta_{h_\gamma}) = \mathbb{R} + \rho^{2\gamma/\beta} H_b^2(Q, dA_{h_\gamma}).$$

The map Φ_γ is defined in a slightly different way here. Let $\mathcal{D}(M, dA_{h_\gamma})$ be the closure of the orthogonal complement of $\mathbb{R}$ (the constant functions) in $L^2(M, dA_{h_\gamma})$. Then Φ_γ is defined as

$$\begin{aligned} \Phi_\gamma \quad &: \quad \mathbb{R} \oplus \mathcal{D}(M, dA_{h_\gamma}) \to \mathbb{R} \oplus \mathcal{D}(M, dA_{g_\beta}), \quad (u_0, v) \mapsto (u_0, e^{\sigma_\gamma} v) \\ \Phi_\gamma^{-1} \quad &: \quad \mathbb{R} \oplus \mathcal{D}(M, dA_{g_\beta}) \to \mathbb{R} \oplus \mathcal{D}(M, dA_{h_\gamma}), \quad (u_0, v) \mapsto (u_0, e^{-\sigma_\gamma} v). \end{aligned} \tag{5.2}$$

Φ_γ is then an isometry from its domain onto its image. The relationship between the domains of the operators is given by the following:

Proposition 9. *For all $\gamma \in [\beta, \pi)$, we have*

$$\Phi_\gamma\left(\mathrm{Dom}(\Delta_{h_\gamma})\right) \subseteq \mathbb{R} + \rho^{2\gamma/\beta} H_b^2(M, dA_{g_\beta}).$$

Moreover,

$$\Phi_\gamma\left(\mathrm{Dom}(\Delta_{h_\gamma})\right) \subset \Phi_{\gamma'}\Big(\mathrm{Dom}(\Delta_{h_{\gamma'}})\Big), \quad \gamma' < \gamma.$$

Proof. The statements above for the case of a surface with an isolated conical singularity follow in the same way as the proof of Proposition 5 using the definition of Φ_γ given in equation (5.2). $\square$

Proof of Proposition 1. The proof in this case is quite similar to the proof of Theorem 1. First, we have

$$\frac{\partial}{\partial\gamma}\mathrm{Tr}_{L^2(M,h_\gamma)}(e^{-t\Delta_{h_\gamma}} - P_{\mathrm{Ker}(\Delta_{h_\gamma})})\bigg|_{\gamma=\alpha} = \frac{\partial}{\partial\gamma}\ \mathrm{Tr}_{L^2(M,g)}(e^{-t\Delta_{h_\gamma}})\bigg|_{\gamma=\alpha}.$$

By Proposition 6 using the same argument as in the proof of Theorem 1, we have

$$\frac{\partial}{\partial\gamma}\ \mathrm{Tr}_{L^2(M,g)}(e^{-t\Delta_{h_\gamma}})\bigg|_{\gamma=\alpha} = -t\ \mathrm{Tr}_{L^2(M,g)}(\dot{\Delta}_g e^{-t\Delta_g}) = 2t\ \mathrm{Tr}_{L^2(M,g)}\left((\delta\sigma_\alpha)\Delta_g e^{-t\Delta_g}\right).$$

Since we also have

$$\frac{\partial}{\partial t}\mathrm{Tr}_{L^2(M,g)}((\delta\sigma_\alpha)e^{-t\Delta_g}) = -2t\frac{\partial}{\partial t}\mathrm{Tr}_{L^2(M,g)}\Big((\delta\sigma_\alpha)\big(e^{-t\Delta_g} - P_{\mathrm{Ker}(\Delta_g)}\big)\Big),$$

the same integration by parts argument gives

$$\frac{\partial}{\partial\gamma}\zeta_{\Delta_{h_\gamma}}(s)\Big|_{\gamma=\alpha} = \frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}_{L^2(M,g)}\Big(2(\delta\sigma_\alpha)\big(e^{-t\Delta_g} - P_{\mathrm{Ker}(\Delta_g)}\big)\Big)dt.$$

As in the proof of Theorem 1, for big values of t the trace can be integrated because we have an estimate of the form

$$\begin{aligned}|\mathrm{Tr}((\delta\sigma_\alpha)\big(e^{-t\Delta_g} - P_{\mathrm{Ker}(\Delta_g)}\big))| &\le \|(\delta\sigma_\alpha)e^{-\frac{1}{2}\Delta_g}\big(e^{-(t-\frac{1}{2})\Delta_g} - P_{\mathrm{Ker}(\Delta_g)}\big)\|_1 \\ &\le \|(\delta\sigma_\alpha)e^{-\frac{1}{2}\Delta_g}\|_1\|e^{-(t-\frac{1}{2})\Delta_g} - P_{\mathrm{Ker}(\Delta_g)}\|_{L^2(M,g)} \ll e^{-c'_\alpha t}\end{aligned}$$

for some $c'_\alpha > 0$, where $\|\cdot\|_1$ denotes the trace norm.

Then just like the classical deduction of Polyakov's formula described at the beginning of §2, the variation of $\zeta'_{\Delta_g}(s)$ at $s=0$ is given by the constant term in the asymptotic expansion as $t \downarrow 0$ of

$$\mathrm{Tr}_{L^2(M,g)}\left(2\mathcal{M}_{\delta\sigma_\alpha}(e^{-t\Delta_g} - P_{\mathrm{Ker}(\Delta_g)})\right).$$

The operators $\mathcal{M}_{\delta\sigma_\gamma}H_\gamma e^{-tH_\gamma}$, $\Phi_\gamma(\delta\Delta_{h_\gamma})\Phi_\gamma^{-1}e^{-tH_\gamma}$, and $\Phi_\gamma\Delta_{h_\gamma}(\delta\sigma_\gamma)\Phi_\gamma^{-1}e^{-tH_\gamma}$ were shown to be trace class in Lemma 4 for all γ considered here. This completes the proof. $\square$

6. Determinant of the Laplacian on rectangles

In this section we prove Proposition 2. Consider a rectangle of width $1/L$ and length L. The spectrum of the Euclidean Laplacian on this rectangle with Dirichlet boundary condition can easily be computed using separation of variables, and it is

$$\left\{\frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{w^2}\right\}_{m,n\in\mathbb{N}}.$$

Consequently the spectral zeta function has the following expression:

$$\begin{aligned}\zeta_L(s) &= \sum_{m,n\in\mathbb{N}}\left(\frac{1}{\pi^2m^2/L^2 + \pi^2n^2L^2}\right)^s \\ &= (\pi)^{-2s}\sum_{m,n\in\mathbb{N}}\frac{1}{|L|^{2s}|mz+n|^{2s}}, \qquad z = \frac{i}{L^2}.\end{aligned}$$

Proof of Proposition 2. We would like to use the computations in [30, p. 204–205], and so we relate the above expression for the zeta function to the corresponding expression in [30] for the torus by

$$\zeta_L(s) = \frac{(\pi)^{-2s}}{2}\left(\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus(0,0)}\frac{1}{|L|^{2s}|mz+n|^{2s}} - 2L^{-2s}\sum_{n\in\mathbb{N}}\frac{1}{n^{2s}} - 2L^{2s}\sum_{m\in\mathbb{N}}\frac{1}{m^{2s}}\right).$$

By [30, p. 204–205],

$$G(s) := \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus(0,0)}\frac{1}{|L|^{2s}|mz+n|^{2s}}$$

satisfies

$$G(0) = -1, \quad G'(0) = -\frac{1}{12}\log\left((2\pi)^{24}\frac{(\eta(z)\bar{\eta}(z))^{24}}{(L)^{24}}\right),$$

where η is the Dedekind η function. Consequently,

$$\zeta_L(s) = \frac{1}{2\pi^{2s}} \left(G(s) - 2L^{-2s}\zeta_R(2s) - 2L^{2s}\zeta_R(2s) \right),$$

where $\zeta_R(s)$ denotes the Riemann zeta function $\zeta_R(s) = \sum_{n\in\mathbb{N}} n^{-s}$. Since the Riemann zeta function satisfies

$$\zeta_R(0) = -\frac{1}{2}, \quad \zeta_R'(0) = -\log\sqrt{2\pi},$$

we compute

$$\begin{aligned} \zeta_L'(0) &= \frac{1}{2}G'(0) - \log\pi + 2\log(2\pi) = -\log\left(\frac{2\pi|\eta(z)|^2}{L}\right) - \log\pi + 2\log(2\pi) \\ &= \log(2) - \log(|\eta(z)|^2/L). \end{aligned}$$

Consequently we obtain the formula for the determinant

$$\det\Delta_L = e^{-\zeta_L'(0)} = \frac{|\eta(z)|^2}{2L} = \frac{|\eta(i/L^2)|^2}{2L}.$$

Let

$$f(L) := \zeta_L'(0) = -2\log(\eta(i/L^2)) + \log(L) + \log(2).$$

Since $\eta(i/L^2) \to 0$ as $L \to \infty$ (c.f. [30] p. 206), it follows that

$$f(L) \to +\infty \text{ as } L \to \infty. \tag{6.1}$$

We use the following identity from [15, p. 12] for

$$\log\eta(i/y) - \log\eta(iy) = \frac{1}{2}\log(y), \quad y \in \mathbb{R}^+$$

to obtain

$$-i\frac{\eta'(i/y)}{\eta(i/y)y^2} - i\frac{\eta'(iy)}{\eta(iy)} = \frac{1}{2y} \implies 4\eta'(i) = i\eta(i).$$

This shows that

$$f'(L) = \frac{4i\eta'(i/L^2)}{\eta(i/L^2)L^3} + \frac{1}{L} \implies f'(1) = \frac{4i\eta'(i) + \eta(i)}{\eta(i)} = 0. \tag{6.2}$$

Since $\frac{d}{dL}\det\Delta_L = \left(\frac{d}{dL}\log(\det\Delta_L)\right)\det\Delta_L$, and $\det\Delta_L > 0$, we have that

$$\frac{d}{dL}\det\Delta_L\bigg|_{L=1} = 0.$$

Finally, we note that the spectrum of the rectangle with dimensions $L \times 1/L$ is identical to that of the rectangle with dimensions $1/L \times L$, and therefore it suffices to consider $L \geq 1$. Consequently, by (6.1) and (6.2) it follows that $f(L)$ is minimized at $L = 1$. This can be compared to similar results obtained by Chu for n-dimensional flat tori [9]. We conclude that the determinant of the Dirichlet Laplacian is uniquely maximized among all rectangles with fixed area by the square, and the determinant tends to zero as rectangles collapse to a line. □

Concluding remarks. Isospectral polygonal domains are known to exist [13], and one can construct many examples by folding paper [7]. A natural question is: how many polygonal domains may be isospectral to a fixed polygonal domain? Osgood, Phillips and Sarnak used the zeta-regularized determinant to prove that the set of isospectral metrics on a given surface of fixed area is compact in the smooth topology [31]. Can one generalize this result in a suitable way to domains with corners? Is it possible to define a flow, as [30] did, which deforms any initial n-gon towards the regular one over time and increases the determinant? How large is the set of isospectral metrics on a surface with conical singularities? These and further related questions will be the subject of future investigation and forthcoming work.

References

[1] K. Akutagawa, G. Carron, and R. Mazzeo, *The Yamabe problem on stratified spaces,* Geom. Funct. Anal. 24 (2014) 1039–1079.

[2] P. Albin, C.L. Aldana and F. Rochon, *Ricci flow and the determinant of the Laplacian on non-compact surfaces*, Comm. in P. D. E., vol. 38, no. 4, (2013), 711-749.

[3] C.L. Aldana, *Asymptotics of relative heat traces and determinants on open surfaces of finite area,* Ann. Glob. Anal. Geom., August 2013, Volume 44, Issue 2, pp 169-216.

[4] O. Alvarez, *Theory of strings with boundaries: fluctuations, topology and quantum geometry,* Nuclear Phys. B 216, no. 1, (1983), 125–184.

[5] E. Aurell and P. Salomonson, *On functional determinants of laplacians in polygons and simplicial complexes,* Commun. Math. Phys. 165, (1994), 233–259.

[6] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory,* Nucl. Phys. B 241, no. 2, (1984), 333–380.

[7] S. J. Chapman, *Drums that sound the same,* Amer. Math. Monthly, 102, (1995), 124–138.

[8] J. Cheeger, *Spectral Geometry of Singular Riemannian Spaces,* J. Diff. Geom. 18 (1983) 575–657.

[9] P. Chu, *Height of flat tori,* Proc. of the A. M. S., vol. 125, no. 3 (1997), 723–730.

[10] E. B. Davies, *Non-Gaussian aspects of heat kernel behaviour,* J. London Math. Soc. vol. 55, no. 2, (1997), 105–125.

[11] J. Gil, T. Krainer, and G. Mendoza, *Resolvents of elliptic cone operators,* Journal of Functional Analysis, 241, (2006), 1–55.

[12] P. Gilkey, The spectral geometry of a Riemannian manifold. J. Differential Geom. Volume 10, Number 4 (1975), 601-618.

[13] C. Gordon, D. L. Webb, and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds,* Invent. Math. 110, no. 1, (1992), 1–22.

[14] P. Grisvard, *Singularities in boundary value problems,* Masson, (1992).

[15] G. Köhler, *Eta Products and Theta Series Identities,* Springer Monographs in Mathematics, Springer-Verlag, Berlin, (2011).

[16] S.W. Hawking, *Zeta function regularization of path integrals in curved spacetime,* Comm. Math. Phys. 55, no. 2, (1977), 133–148.

[17] L. Hillairet and A. Kokotov, *Krein Formula and S-Matrix for Euclidean Surfaces with Conical Singularities,* Journal of Geometric Analysis, vol. 23, no. 3, (2013), 1498–1529.

[18] A. Kokotov, *Compact polyhedral surfaces of arbitrary genus and determinants of Laplacians,* Proc. Amer. Math. Soc. 141, no. 2, (2013), 725–735.

[19] A. Kokotov and D. Korotkin, *Normalized Ricci Flow on Riemann Surfaces and Determinant of Laplacian.* Lett. Math. Phys. 71 (2005), no. 3, 241-242.

[20] O. Ladyzhenskaya and N.N Ural'tseva, *Linear and quasilinear elliptic equations,* Academic Press, New York-London 1968.

[21] N.N. Lebedev *Special functions and their applications,* Englewood Cliffs, N. J. : Prentice-Hall, inc., 1965

[22] M. Lesch, *Operators of Fuchs Type, Conical Singularities, and Asymptotic Methods,* Teubner-Texte Math., vol. 136, Teubner, Stuttgart, (1997).

[23] P. Loya, P. McDonald, J. Park, *Zeta regularized determinants for conic manifolds.* J. Funct. Anal. 242 (2007), no. 1, 195- 229.

[24] Y.L. Luke *The special functions and their approximations, Vol II,* Volume 53 in Mathematics in Science and Engineering. Academic Press, 1969, New York and London.

[25] R. Mazzeo, *Elliptic theory of differential edge operators I,* Comm. Partial Diff. Eq., 16, no. 10, (1991), 1615–1664.

[26] R. Mazzeo and G. Montcouquiol, *Infinitesimal rigidity of cone-manifolds and the Stoker problem for hyperbolic and Euclidean polyhedra,* J. Diff. Geom. 87, no. 3, (2011), 525–576.

[27] R. Mazzeo and J. Rowlett, *A heat trace anomaly on polygons,* to appear in Math. Proc. of the Cambridge Philosophical Society.

[28] G. Mendoza, *Zeta functions of elliptic cone operators,* Partial Differential Equations and Spectral Theory, Operator Theory: Advances and Applications, Springer, (2011), 291–320.
[29] W. Müller, *Relative zeta functions, relative determinants and scattering theory,* Comm. Math. Phys. 192, no. 2, (1998), 309–347.
[30] B. Osgood, R. Phillips, and P. Sarnak, *Extremals of Determinants of Laplacians,* Journ. Funct. Anal., 80, (1988), 148–211.
[31] B. Osgood, R. Phillips and P. Sarnak, *Compact isospectral sets of surfaces.* J. Funct. Anal. 80 (1988), 212–234.
[32] A. M. Polyakov, *Quantum geometry of bosonic strings,* Physics Letters B, vol. 103, no. 3, (1981), 207–210.
[33] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds,* Advances in Math. 7, (1971), 145–210.
[34] D.A. Sher. *Conic degeneration and the determinant of the Laplacian.* arXiv:1208.1809.
[35] M. Spreafico, *Zeta function and regularized determinant on a disc and on a cone.* J. Geom. Phys. 54 (2005), no. 3, 355-371.
[36] M. van den Berg and S. Srisatkunarajah, *Heat equation for a region in $\mathbb{R}^2$ with a polygonal boundary,* J. London Math. Soc. vol. 37, no. 2, (1988), 119–127.
[37] G. N. Watson, *A treatise on the theory of Bessel functions,* Cambridge University Press, (1922).
[38] H. Weyl, *Singuläre Integralgleichungen,* Math. Ann. 66, no. 3, (1908), 273–324.

University of Luxembourg, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg
E-mail address: clara.aldana@uni.lu

Department of Mathematics, Chalmers University of Technology, 41296 Gothenburg, Sweden
E-mail address: julie.rowlett@chalmers.se